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Introduction to approximation theory and wavelets with applications to the signal processing

A short preface

This is an introductory course on applications of the approximation theory to the construction of wavelets, signal processing and image compression.

We start with some concepts of the classical approximation theory such as uniform approximation, Chebyshev sets, etc.

We show disadvantages of the standard Fourier system and motivate construction of new orthonormal bases: wavelets, frames, etc. Then we consider in detail the first wavelets: Haar system an its "dual" Shannon-Kotelnikov system, passing to more advanced Meyer systems, and finally, to compactly-supported wavelets, including Daubechies wavelets. The general construction of wavelets is introduced by the fundamental concept of multiresolution analysis, which motivates refinement functional equations as starting point. This allows us to derive the properties of compactly-supported wavelets, such as approximations, local and global regularity and relation to fractals.

Program of the course

1. Best uniform approximation. Compactness and existence of best approximation. Chebyshev's theorem and de la Vallee Poussin's estimates.

2. Convexity of Chebyshev sets in $\mathbb{R}^n$ and on the normed plane.

3. Fourier series and Fourier transform. Time and frequency domains. Some applications of orthonormal systems (signal and image processing, numerical methods in PDE). Fast Fourier transform. Noise sensitivity.

4. Haar systems on a segment and on a line. Regularity, approximation properties and applications. Shannon-Kotelnikov wavelets. The sampling theorem.

5. The general construction of wavelets. Multiresolution analysis. Scaling (refinable) function as the "mother" of a wavelet system. Fast computing of wavelet coefficients. The cascade algorithm. Examples. Time-frequency localization. Uncertainty constant. Wavelets of Meyer.

6. Advantages of compact support. Do compactly-supported smooth wavelets exist ? Grippenberg's theorem. Refinement equations. Mask and symbol. Existence, uniqueness and formula for the solution.

7. Daubechies wavelets. Construction and main properties. Fractal nature of wavelets.