Higher order Goh conditions

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joint work with

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Seminar in Geometric Control Theory
Moscow, 16th February 2022

Brief history

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Today's results origin from a meeting of these two traditions.

Example, part 1

In the manifold $M = \mathbb{R}^3$ consider the two vector fields

$$f_1 = \frac{\partial}{\partial x}, \quad f_2 = (1-x)\frac{\partial}{\partial y} + x^n \frac{\partial}{\partial z},$$

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$$\dot{\gamma} = u_1 f_1(\gamma) + u_2 f_2(\gamma), \quad u = (u_1, u_2) \in L^2([0,1]; \mathbb{R}^2) \quad \text{“control”}.$$

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The length of γ is

$$L(\gamma) = \int_0^1 |u| dt.$$

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In this talk, I will try to explain WHY.

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Bad case. When γ is singular the situation is confused because there are many

– examples of singular extremals that are minimizing (all smooth);
– examples of nonsmooth singular extremals, we do not know if minimizing or not.

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Definition. The control $u \in X$ is *singular* if the differential $d_u F : X \to T_{F(u)} M$ is not surjective.

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1) The bracket $[f_i, f_j]$ appears in the computation for the Hessian $\mathscr{D}^2_u F$ of the end-point map F.

2) The proof of ii) goes as follows:

$$\langle \lambda, [f_i, f_j](\gamma) \rangle \neq 0 \quad \Rightarrow \quad \mathrm{index}(\mathscr{D}^2_u F) = \infty \quad \overset{strictly}{\Rightarrow} \quad \mathscr{F} \text{ open}$$

Above, $\mathscr{F} = (F, L) =$ extended end-point map with $L =$ length.

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Definition. The singular point $u = 0$ is *strictly* singular when

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The L component is quotiented out by π.

Multi-linear differentials

Let $F : X \to \mathbb{R}^m$ be smooth, $n \in \mathbb{N}$ and $v_1, \ldots, v_n \in X$.

Definition. We define the nth-order multilinear derivative of F at $u = 0$

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Comment. When $n = 2$ we are defining the usual intrinsic Hessian

$$\mathscr{D}_0^2 F : \ker(d_0 F) \to \mathrm{coker}(d_0 F).$$

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b) n even and there are $v^{\pm} \in \mathrm{dom}(\mathscr{D}_0^n F)$ such that $\mathscr{D}_0^n F(v^{\pm}) = \pm 1$;

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a) n odd and there is $v \in \mathrm{dom}(\mathscr{D}_0^n F)$ such that $\mathscr{D}_0^n F(v) \neq 0$; OR,

b) n even and there are $v^{\pm} \in \mathrm{dom}(\mathscr{D}_0^n F)$ such that $\mathscr{D}_0^n F(v^{\pm}) = \pm 1$;

Then $\mathscr{D}_0^n F$ is regular.

Example, part 3

Consider again $M = \mathbb{R}^3$ with

$$f_1 = \frac{\partial}{\partial x}, \quad f_2 = (1-x)\frac{\partial}{\partial y} + x^n \frac{\partial}{\partial z}.$$

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b) n even: we have $\mathscr{D}_u^n F \geq 0$. This "coercivity" is compatible with the local minimality of γ.

Taylor expansion, part 1

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Pull back via optimal flow. Let u be the control of γ. For $i = 1, \dots, d$ we define the time-dependent vector-fields

$$g_i^t = g_i^{u,t} = (P_t^1)_* f_i = \mathrm{Ad}\left(\overrightarrow{\exp} \int_1^t f_{u(\tau)} d\tau\right) f_i.$$

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Localization of controls. For $t_0 \in [0, 1)$ and $s > 0$ small we let

$$v_{t_0,s}(t) = v\left(\frac{t - t_0}{s}\right) \chi_{[t_0, t_0+s]}(t), \quad t \in [0, 1].$$

Taylor expansion, part 2

Definition of V_{n-1}. We say that $v = (v^1, \dots, v^d) \in L^1([0,1]; \mathbb{R}^d)$ belongs to V_{n-1} if for any $1 \le h \le n-1$

$$\int_{\Sigma_h} v^{j_1}(t_1) \dots v^{j_h}(t_h) dt_1 \dots dt_h = 0, \quad j_1, \dots, j_k \in \{1, \dots, d\},$$

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Theorem. If $v \in V_{n-1}$ then

$$D_u^n F(v_{t_0,s}, *) = s^n \int_{\Sigma_n} [g^{t_0}_{v(t_n)}, [\dots, [g^{t_0}_{v(t_2)}, g^{t_0}_{v(t_1)}] \dots]] dt + O(s^{n+1}).$$

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Corollary. We have the necessary condition

$$(\dagger) \quad 0 = \mathscr{G}_{t_0}(v) = \sum_{\alpha} \langle \lambda, [g_{\alpha_n}^{t_0}, [\dots, [g_{\alpha_2}^{t_0}, g_{\alpha_1}^{t_0}] \dots]] \rangle \int_{\Sigma_n} v^{\alpha_n}(t_n) \dots v^{\alpha_1}(t_1) dt,$$

where $\lambda = \lambda(1)$ is the covector at the end-point originating the adjoint curve.

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Conclusion. For each fixed multi-index α we have

$$\langle \lambda, [g^{t_0}_{\alpha_n}, [\ldots, [g^{t_0}_{\alpha_2}, g^{t_0}_{\alpha_1}] \ldots]] \rangle = 0.$$

These are the Goh conditions of order n.

You find the preprint on arxiv:

Boarotto, Monti, Socionovo
Higher order Goh conditions for singular extremals of corank 1

Or write me: monti@math.unipd.it

Thank you for your patient attention.