## Multidirectional Mean Value Inequalities and Dynamic Optimization

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#### **Ulisse Dini (1845 – 1918)**



#### Dini directional derivative for $f : \mathbb{R}^n \to (-\infty, +\infty]$

$$\underline{D} f(x;v) := \liminf_{t \downarrow 0, w \to v} \frac{f(x+tw) - f(x)}{t}$$

Envelopes of parametric families  $\{f_{\gamma}(x)\}_{\gamma \in \Gamma}$ 

$$f(x) = \sup_{\gamma \in \Gamma} f_{\gamma}(x), \quad h(x) = \inf_{\gamma \in \Gamma} f_{\gamma}(x)$$

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(sup-envelope and inf-envelope) k-th largest eigenvalue of a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ :

$$\lambda_k(A) = \max_{X \in S(k,n)} \operatorname{tr}(X^T A X) - \max_{X \in S(k-1,n)} \operatorname{tr}(X^T A X)$$

where the Stiefel manifold

$$S(k,n) := \{ X \in \mathbb{R}^{n \times k} : X^T X = I_k \}$$

#### Nonsmooth control Lyapunov functions: Finite dimensional nonlinear control system:

 $\dot{x}(t) = f(x(t), u(t)), \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{U}$ 

*x* - state vector, *u* - control (null-) asymptotic controllability: for each  $x_0 \exists$  "open loop" control  $u : [0, +\infty) \rightarrow \mathbb{U}$ 

 $x(t;x_0,u) \to 0$ 

feedback stabilizer :

 $k : \mathbb{R}^n \to \mathbb{U}$  s.t. all trajectories of the system

$$\dot{x} = f(x, k(x))$$

 $x(t) \rightarrow 0$  in some uniform and stable manner

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$$\langle V'(x), f(x, k(x)) \rangle \le -W(x) < 0 \quad \forall \ x \ne 0$$

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$$\min_{v \in \operatorname{co} f(x,U)} \underline{D} V(x;v) \le -W(x) < 0 \quad \forall \ x \ne 0$$

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Affirmative answer: Clarke, Ledyaev, Sontag, Subbotin 1996

Asymptotic controllability

How to characterize infinitesimal properties of non-differentiable (nonsmooth) functions

Dini directional derivative for  $f : \mathbb{R}^n \to (-\infty, +\infty]$ 

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Subgradient  $\zeta \in \partial f(x)$  definition:  $\exists$  smooth  $g : \mathbb{R}^n \to \mathbb{R}$  such that  $\zeta = g'(x)$ , function f(y) - g(y)attains local minimum at x

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- f(y) is convex,  $g(y) = a + \langle \zeta, y \rangle$  is affine (subgradient in convex analysis sense)
- f is lower semicontinuous,  $g(y) = a + \langle \zeta, y \rangle \sigma \|y x\|^2$  (proximal subgradient)
- f is lower semicontinuous, g(y) is differentiable (smooth) ( "Fréchet" subgradient)





Foundations of nonsmooth analysis:

Clarke, Danskin, Demyanov, Ioffe, Mordukhovich, Moreau, Pschenichny, Rockafellar, Rubinov, Subbotin, Vinter, Borwein

What is relation between subgradients and directional derivatives?

What is relation between subgradients and directional derivatives? Obvious : if  $\zeta \in \partial f(x)$  then (use  $g(y) - g(x) \le f(y) - f(x) \forall y$  near x)

 $\langle \zeta, v \rangle \le \underline{D}f(x; v) \quad \forall v$ 

What is a relation between subgradients and directional derivatives? Obvious : if  $\zeta \in \partial f(x)$  then

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Non-obvious :

What is a relation between subgradients and directional derivatives?

**Obvious :** if  $\zeta \in \partial f(x)$  then

$$\langle \zeta, v \rangle \le \underline{D}f(x; v) \quad \forall v$$

Non-obvious : Subbotin's Theorem

**THEOREM:** Let  $V \subset \mathbb{R}^n$  be nonempty, convex, and bounded,  $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$  be lower semicontinuous. For  $\forall r$  such that

 $\inf_{v \in V} \underline{D}f(x; v) > r$ 

and  $\forall \varepsilon > 0 \exists z \in x + \varepsilon B_{\mathbb{R}^n}$  with  $|f(z) - f(x)| < \varepsilon$  and  $\zeta \in \partial_F f(z)$  such that

 $\min_{v \in V} \langle \zeta, v \rangle > r$ 

Applications of Subbotin's theorem:

- equivalence of Subbotin's minmax solutions (1980) of Hamilton-Jacobi equations and viscosity solutions
- for control Lyapunov function: condition

$$\min_{v \in \operatorname{co} f(x,U)} \underline{D} V(x;v) \le -W(x) < 0 \quad \forall \ x \ne 0$$

is equivalent to the condition

 $\min_{u \in U} \langle V', f(x, u) \rangle \le -W(x) < 0 \quad \forall \ V' \in \partial V(x), \ x \neq 0$ 

How to prove Subbotin's theorem?

## **Multidirectional Mean Value Inequalities**

Classical (Lagrange) Mean value Theorem:  $\exists z \in [x^0, x]$ 

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Multidirectional Mean Value Inequality (Clarke and Ledyaev) for differentiable f, convex, closed and bounded set  $X \subset \mathbb{R}^n$ :  $\exists z \in [x^0, X] := \operatorname{co}(\{x^0\} \cup X)$ 

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$$\min_{x \in X} f(x) - f(x^0) \le \langle f'(z), x - x^0 \rangle \quad \forall \ x \in X$$

Proof by using "clever function":

$$h(t,y) := f(x^0 + t(y - x^0)) - f(x^0) - rt, \quad y \in X, \ t \in [0,1]$$

where

$$r := \min_{x \in X} f(x) - f(x^0)$$

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$$\min_{x \in X} f(x) - f(x^0) \le \langle f'(z), x - x^0 \rangle \quad \forall \ x \in X$$

General case (Clarke and Ledyaev 1994): lower semicontinuous  $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}, \mathbb{E}$  - Hilbert space,  $X \subset \mathbb{E}$  closed, convex and bounded

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**THEOREM:** For  $\forall r \text{ and } \varepsilon > 0$   $r < \sup_{\delta > 0} \inf_{x \in X + \delta B} f(x) - f(x^0)$   $\exists z \in [x^0, X] + \varepsilon B \text{ and } \zeta \in \partial_P f(z)$   $r < \langle \zeta, x - x^0 \rangle \quad \forall x \in X$ and  $f(z) \le \inf_{x \in X} f(x) + \max\{0, r\} + \varepsilon$ . General case (Clarke and Ledyaev 1994): lower semicontinuous  $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}, \mathbb{E}$  - Hilbert space,  $X \subset \mathbb{E}$  closed, convex and bounded

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Generalizations: Aussel, Corvellec, Lassonde, Clarke, Radulescu, Ivanov, Zlateva, Zhu et al.

## **Multidirectional Mean Value Inequalities**

#### Generalization of Subbotin's theorem for smooth Banach spaces $\mathbb{E}$ :

**THEOREM:** Let  $V \subset \mathbb{E}$  be nonempty, convex, and bounded,  $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$  be lower semicontinuous. For  $\forall r$  such that

 $\inf_{v \in V} \underline{D}^w f(x;v) > r$ 

and  $\forall \varepsilon > 0 \exists z \in x + \varepsilon B_{\mathbb{R}^n}$  with  $|f(z) - f(x)| < \varepsilon$  and  $\zeta \in \partial_F f(z)$  such that

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## **Multidirectional Mean Value Inequalities**

#### Generalization of Subbotin's theorem for smooth Banach spaces $\mathbb{E}$ :

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#### **Implicit Multifunction Theorems**

Let F(x,p) = G(x,p) - K where  $G : \mathbb{E} \times \mathbb{P} \to \mathbb{Y}$  smooth function,  $K \subset \mathbb{Y}$  - closed convex cone. Implicit multifunction

$$X(p) := \{ x \in \mathbb{X} : G(x, p) \in K \text{ or } 0 \in F(x, p) \}$$

Let  $0 \in F(x_0, p_0)$  and  $G_x(x_0, p_0) \mathbb{X} - K = \mathbb{Y}$ . Then for  $\forall x, p$  near  $x_0, p_0$ 

 $X(p) \neq \emptyset, \quad d(x, X(p) \le kd(0, F(x, p))$ 

Let  $f : \mathbb{E} \to \mathbb{R}$  be lower semicontinuous

 $S := \{ x \in \mathbb{E} : f(x) \le 0 \}$ 

Assumptions:  $\exists \Delta > 0$  such that  $\forall x \notin S$ 

 $\|\zeta\| > \Delta \quad \forall \, \zeta \in \partial_F f(x)$ 

If  $S = \emptyset$  then for any  $\rho > 0$  by the multidirectional MVI  $\exists \zeta \in \partial_F f(z)$ 

 $\inf_{x \in x^0 + \rho B} f(x) \le f(x^0) + \min_{x \in x^0 + \rho B} \langle \zeta, x - x^0 \rangle = f(x^0) - \rho \|\zeta\| < f(x^0) - \rho \Delta$ 

Contradiction if  $\rho > f(x^0)/\Delta!$  Thus,  $(x \in x^0 + \rho B) \cap S \neq \emptyset$  and

 $d_S(x^0) \leq \frac{f(x^0)}{\Delta}$  (METRIC REGULARITY and SOLVABILITY)

Let F(x,p) be multifunction ,  $F(x,p) \subset \mathbb{Y}$ 

Let F(x, p) be multifunction ,  $F(x, p) \subset \mathbb{Y}$ Consider implicit multifunction X(p)

$$X(p) := \{ x \in \mathbb{E} : 0 \in F(x, p) \}$$

define  $f(x,p) := d_{F(x,p)}(0)$ and obtain "Implicit Multifunction Theorems" (1999) Ledyaev and Zhu even earlier 'Theorems on Implicitly-Defined Multi-Valued Mappings" (1984) Ledyaev

Representation of subgradients of sup- and inf-envelopes of parametric family  $\{f_{\gamma}(x)\}_{\gamma\in\Gamma}$ 

$$f(x) := \sup_{\gamma \in \Gamma} f_{\gamma}(x), \quad h(x) := \inf_{\gamma \in \Gamma} f_{\gamma}(x)$$
$$f_{\gamma}(x) := \gamma \sin(\gamma x) - \frac{\gamma^2}{4}, \quad \gamma \in [-2, 2]$$



6/11

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Let  $\zeta \in \partial_F f(x)$ . Then,  $\forall \varepsilon > 0$  and  $\forall \delta > 0$ ,  $\exists$  convex coefficients  $\{\alpha_k\}$ , pairs  $(x_k, \gamma_k) \in G_{\delta}(x)$ ,  $k = 1, \dots, K$ , such that  $\zeta \in \sum_{k=1}^K \alpha_k \partial_F f_{\gamma_k}(x_k) + \varepsilon B$ 

If, in addition, the functions  $f_{\gamma}$  are continuous then the set  $G_{\delta}(x)$  can be replaced by the smaller set  $G_{\delta}^{0}(x)$ .

 $G_{\delta}(x) := \{(z,\gamma) \in \mathbb{X} \times \Gamma : z \in [x,y] + \delta B \exists y \in x + \delta B, f_{\gamma}(y) \ge f(x) - \delta \}$ 

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In the case  $x \in \mathbb{R}^n$  then [K = n + 1!]

Applications for manifolds of nonpositive curvature: Jung theorem's generalization on radius of circumscribed ball

$$R(\mathbb{S}) \le \frac{1}{2} \sqrt{\frac{2n}{n+1}} \ D(\mathbb{S})$$

Helly theorem generalization for intersection of convex sets on manifolds of nonpositive curvature

#### Ledyaev, Treiman and Zhu

**THEOREM:** Let  $\mathcal{K} = \{K_{\gamma}\}_{\gamma \in \Gamma}$  be a family of convex sets and  $\Gamma$  is finite, or all  $K_{\gamma}$  are compact. Then if any n + 1 sets from  $\mathcal{K}$  have a common point, all sets  $K_{\gamma}$  have a common point.

Representation of subgradients of sup- and inf-envelopes of parametric family  $\{f_{\gamma}(x)\}_{\gamma \in \Gamma}$ 

$$f(x) := \sup_{\gamma \in \Gamma} f_{\gamma}(x), \quad h(x) := \inf_{\gamma \in \Gamma} f_{\gamma}(x)$$

Let  $\zeta \in \partial_F h \Longrightarrow \exists$  function  $\varphi(\varepsilon)$  s.t.  $\varphi(\varepsilon) \downarrow 0$  as  $\varepsilon \downarrow 0$  and  $\forall \varepsilon > 0$ , and  $\forall (\gamma_{\varepsilon}, x_{\varepsilon})$  with

$$x_{\varepsilon} \in x + \varepsilon B$$
 and  $f_{\gamma_{\varepsilon}}(x_{\varepsilon}) < h(x) + \varepsilon$ 

there exist  $z \in x + 2\sqrt{\varepsilon}B$  s.t.

 $f_{\gamma_{\varepsilon}}(z) < h(x) + O(\sqrt{\varepsilon})$ 

and

$$\zeta \in \partial_F f_{\gamma_{\varepsilon}}(z) + \varphi(\varepsilon)B.$$

Application:

- Existence and uniqueness of minimizers!
- For, example (for simplicity  $\mathbb{E}$  is Hilbert):

$$h(x^{\star}) := \inf_{y \in S} (f(y) + \langle x^{\star}, y \rangle)$$

If for some (a.a.)  $x^* \in \mathbb{E}^* \exists \zeta \in \partial_F h(x^*) \neq \emptyset \implies \forall \varepsilon > 0 \exists y_{\varepsilon} \in S$ ,  $f(y_{\varepsilon}) + \langle x^*, y_{\varepsilon} \rangle < h(x^*) + \varepsilon$ 

 $\zeta \in y_{\varepsilon} + \varphi(\varepsilon)B$ 

 $ightarrow \varepsilon$ -minimizer  $y_{\varepsilon}$  converges to <u>unique</u> minimizer  $y_0 = \zeta$  of the function

$$y \to f(y) + \langle x^\star, y \rangle$$

on bounded S - generalization of Stegall variational principle

Uniqueness of closest points to the set  $S \subset \mathbb{E}$ , for simplicity  $\mathbb{E}$  is Hilbert.

**Consider function** 

$$h(x) := \inf_{y \in S} ||x - y||^2 = d_S^2(x)$$

Let  $\zeta \in \partial_F h(x) \longrightarrow \forall \varepsilon > 0 \exists y_{\varepsilon} \in S$ 

$$||x - y_{\varepsilon}||^2 < h(x) + \varepsilon, \quad \zeta = 2(x - y_{\varepsilon}) + \varphi(\varepsilon)B$$

 $figure y_{\varepsilon}$  converges to  $y_0 = x - \zeta/2$  unique closest to x point in S. More results in

Ledyaev and Treiman "Sub- and Super-gradients of Envelopes, Semicontinuous Closures and Limits of Functions"

Application of multidirectional MVI to deriving optimality conditions for nonsmooth calculus of variations problem

Application of multidirectional MVI to deriving optimality conditions for nonsmooth calculus of variations problem Consider minimizer  $x_*$  of a nonsmooth functional  $J: W^{1,p} \to \mathbb{R} \cup \{+\infty\}$  of the form

$$J(x) = \ell(x(a), x(b)) + \int_{a}^{b} L(t, x(t), \dot{x}(t)) dt,$$

where  $W^{1,p}$  ( $1 \le p \le \infty$ ) denotes the Banach space of absolutely continuous maps  $x : [a, b] \to \mathbb{R}^d$  satisfying  $\dot{x} \in L^p(a, b; \mathbb{R}^n)$ .

Application of multidirectional MVI to deriving optimality conditions for nonsmooth calculus of variations problem Consider minimizer  $x_*$  of a nonsmooth functional  $J: W^{1,p} \to \mathbb{R} \cup \{+\infty\}$  of the form

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**Difficulties for manifolds – combining global and local!** 

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#### Difficulties for manifolds – combining global and local!

Kipka and Ledyaev "Optimal control of differential inclusions on manifolds" based on

Kipka and Ledyaev "Extension of Chronological Calculus for Dynamical Systems on Manifolds"

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Long history: starting with Clarke, in 1990s important developments Mordukhovich, Rockafellar, loffe, Smirnov, Vinter

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Long history: starting with Clarke, in 1990s important developments Mordukhovich, Rockafellar, Ioffe, Smirnov, Vinter For problems with differential constraints

$$\dot{x}(t) \in F(t, x(t))$$

consider penalized nonsmooth functional with

$$L(t, x, \dot{x}) := L_0(t, x, \dot{x}) + kd(\dot{x}, F(t, x))$$

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In this work we generalize a result by Clarke from Clarke" Functional analysis, calculus of variations, and optimal control", Springer-Verlag, 2013. Some assumptions are relaxed and shorter proof based on MVI

**Geometric Control Seminar, MSU, April 22, 2020** – p. 7/11

## **Assumptions and Optimality Conditions**

#### Minimize

$$J(x) = \ell(x(a), x(b)) + \int_{a}^{b} L(t, x(t), \dot{x}(t)) dt,$$

for  $x \in W^{1,p}$  s.t.  $\dot{x}(t) \in V(t)$  a.a. t

V(t) measurable multifunction with closed convex values

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V(t) measurable multifunction with closed convex values ASSUMPTIONS:

- (Controllability)  $\exists \delta_* > 0$  s.t. for a.a.  $t, \dot{x}_*(t) + \delta_* B \subset V(t)$
- Function  $\ell : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$  is lower semicontinuous;
- For a given  $x_* \in W^{1,p}$  there exists  $\varepsilon_* > 0$  such that for some  $k_L \in L^2(a,b;\mathbb{R})$ , for a.a.  $t \in [a,b]$ ,  $\forall x, y \in x_*(t) + \varepsilon B$ , and  $\forall u, v \in V(t)$  $|L(t,x,u) - L(t,y,v)| \le k_L(t) \{ ||x-y||_{\mathbb{R}^d} + ||u-v||_{\mathbb{R}^d} \}$

Type of minimizer:

 $x_*$  is a V-local minimizer for J if  $\exists \varepsilon_* > 0$  s.t.  $\forall x \in W^{1,p}$  satisfying  $\dot{x}(t) \in V(t)$  for a.a.  $t \in [a, b]$  and  $||x - x_*||_{\infty} < \varepsilon_* \longrightarrow J(x) \ge J(x_*)$ .

**THEOREM:** Let  $x_*$  be *V*-local minimizer under assumptions. Then  $\exists$  absolutely continuous  $p : [a, b] \rightarrow \mathbb{R}^d$  which satisfies the Euler inclusion:

 $\dot{p}(t) \in \operatorname{co} \{ w : (w, p(t)) \in \partial_L L(t, x_*(t), \dot{x}_*(t)) \} \quad a.a. \ t \in [a, b],$ 

the transversality conditions:

 $(p(a), -p(b)) \in \partial_L \ell(x_*(a), x_*(b))$ 

Weierstrass condition: a.a.  $t \in [a, b]$ ,  $\forall v \in V(t)$ ,

 $L(t, x_*(t), v) \ge L(t, x_*(t), \dot{x}_*(t)) + \langle p(t), v - \dot{x}_*(t) \rangle.$ 

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Smooth case  $p(t) = L_v(t, x_*(t), \dot{x}_*(t)), \ p'(t) = L_x(t, x_*(t), \dot{x}_*(t))$ 

- Consider linear mappings  $A : \mathcal{V} \to \mathbb{E}$  and  $\psi : \mathcal{V} \to \mathbb{R}$ , where  $\mathcal{V}$  is an arbitrary convex set. In particular, we make the following assumptions on the data  $f, X, \mathcal{V}, A$  and  $\psi$ :
- A1: The function  $f : \mathbb{E} \to \mathbb{R} \cup \{\infty\}$  is lower semicontinuous,  $f(x_0 + A(v_0))$  is finite, and f is bounded from below on the set

$$D_0 := [x_0 + A(v_0), X + A(\mathcal{V})] + \delta B,$$

for some  $\delta > 0$ ;

- **A2**: The set  $X \subset \mathbb{E}$  is closed, bounded, and convex;
- A3: The sets  $A(\mathcal{V})$  and  $\psi(\mathcal{V}) \subset \mathbb{R}$  are bounded.

#### Kipka and Ledyaev

**THEOREM:** Let  $\mathbb{E}$  admits a bump function *b*(globally Lipschitz and  $\beta$ -smooth), Assumptions A1 – A3 hold. Then  $\forall \varepsilon > 0$  and  $\forall \rho$  s.t.

$$\rho < \sup_{\delta > 0} \inf_{x \in X + \delta B, v \in \mathcal{V}} \left\{ f(x + A(v)) - f(x^0 + A(v^0)) + \psi(v) - \psi(v^0) \right\}$$

 $\exists z \in \left[x^0 + A(v^0), X + A(\mathcal{V})\right] + \varepsilon B \text{ and } z^* \in \partial_\beta f(z) \text{ s.t.}$ 

$$\rho < \langle z^*, x - x^0 + A(v) - A(v^0) \rangle + \psi(v) - \psi(v^0)$$

 $\forall x \in X \text{ and } v \in \mathcal{V}.$ 

More general than existing variants of MVI.

Proof is based on the use of bump function instead of square of Hilbert norm in Clarke and Ledyaev 1994

Traditional variation of minimizer:

$$x^{\lambda}(t) = x_*(t) + \lambda y(t), \quad \dot{y}(t) = v(t)$$

Use relaxed controls  $\mu(t)$  instead v(t)

$$\mu(t) = \sum_{i=1}^{m} \alpha_i \delta_{u_i(t)}$$

 $\alpha_i \ge 0$  satisfy  $\sum_{i=1}^m \alpha_i = 1$ 

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 $\alpha_i \ge 0$  satisfy  $\sum_{i=1}^m \alpha_i = 1$ Then relaxed variation

$$x^{\lambda}(t) = x_{*}(t) + \lambda v + \lambda \int_{a}^{t} w(s) \, ds + \int_{a}^{t} \int_{\mathbb{R}^{d}} u \, \mu^{\lambda}(s; du) \, ds$$
$$J^{\lambda} := \ell(x^{\lambda}(a), x^{\lambda}(b)) + \int_{a}^{b} L(t, x^{\lambda}(t), \dot{x}_{*}(t) + \lambda w(t) + u) \mu^{\lambda}(t; du) \, dt$$

where  $\mu$  is a relaxed velocity,  $\mu^{\lambda} := (1 - \lambda)\delta_{u_*(t)} + \lambda \mu$ ,  $v \in \mathbb{R}^d$ , and  $w \in L^{\infty}(a, b; \mathbb{R}^d)$ ,  $u_*(t) = \dot{x}_*(t)$ 

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$$x^{\lambda}(t) = x_*(t) + \lambda v + \lambda \int_a^t w(s) \, ds + \int_a^t \int_{\mathbb{R}^d} u \, \mu^{\lambda}(s; du) \, ds$$
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Approximation: Any  $(x^{\lambda}, J^{\lambda} \text{ can be approximated with arbitrary precision by traditional control <math>u(t)$ ! Controllability is used here for lower semicontinuous  $\ell$ .

Let  $\mathbb{E}$  denote the Banach space  $\mathbb{R}^d \times \mathbb{R}^d \times L^2(a, b; \mathbb{R}^d) \times L^2(a, b; \mathbb{R}^d)$ , define a function  $f : E \to \mathbb{R} \cup \{+\infty\}$  through

$$f(x, y, u, v) = \ell(x_*(a) + x, x_*(b) + y) + \int_a^b \widetilde{L}(t, x_*(t) + u(t), \dot{x}_*(t) + v(t)) dt.$$

Define linear maps  $A: \mathcal{V}(R, \varepsilon) \to \mathbb{E}$  and  $\psi: \mathcal{V}(R, \varepsilon) \to \text{through}$ 

$$A(\mu) = \left(0, \int_{a}^{b} \int_{\mathbb{R}^{d}} u\mu(s, du) \, ds, \int_{a}^{(\cdot)} \int_{\mathbb{R}^{d}} u\mu(s, du) \, ds, 0\right)$$
$$\psi(\mu) = \int_{a}^{b} \int_{\mathbb{R}^{d}} \widetilde{L}(s, x_{*}(s), \dot{x}_{*}(s) + u)\mu(s, du) \, ds$$

#### Then

$$\begin{split} f(x + A(\mu)) &- f(0 + A(\delta_0)) + \psi(\mu) - \psi(\delta_0) = \\ \ell(x^{\lambda}(a), x^{\lambda}(b)) &- \ell(x_*(a), x_*(b)) + \int_a^b \widetilde{L}(t, x^{\lambda}(t), \dot{x}_*(t) + \lambda w(t)) \, dt + \\ &+ \lambda \int_a^b \int_{\mathbb{R}^d} \widetilde{L}(t, x_*(t), \dot{x}_*(t) + u) \, \mu(t, du) - \int_a^b \widetilde{L}(t, x_*(t), \dot{x}_*(t)) \, dt = \\ &= \ell(x^{\lambda}(a), x^{\lambda}(b)) + \int_a^b \int_{\mathbb{R}^d} \widetilde{L}(t, x^{\lambda}(t), \dot{x}_*(t) + \lambda w(t) + u) \mu^{\lambda}(t, du) \, dt - \\ &- J(x_*) - o(\lambda) = J^{\lambda} - J(x_*) - o(\lambda) \ge -o(\lambda). \end{split}$$

#### Then

$$\begin{split} f(x + A(\mu)) &- f(0 + A(\delta_0)) + \psi(\mu) - \psi(\delta_0) = \\ \ell(x^{\lambda}(a), x^{\lambda}(b)) &- \ell(x_*(a), x_*(b)) + \int_a^b \widetilde{L}(t, x^{\lambda}(t), \dot{x}_*(t) + \lambda w(t)) \, dt + \\ &+ \lambda \int_a^b \int_{\mathbb{R}^d} \widetilde{L}(t, x_*(t), \dot{x}_*(t) + u) \, \mu(t, du) - \int_a^b \widetilde{L}(t, x_*(t), \dot{x}_*(t)) \, dt = \\ &= \ell(x^{\lambda}(a), x^{\lambda}(b)) + \int_a^b \int_{\mathbb{R}^d} \widetilde{L}(t, x^{\lambda}(t), \dot{x}_*(t) + \lambda w(t) + u) \mu^{\lambda}(t, du) \, dt - \\ &- J(x_*) - o(\lambda) = J^{\lambda} - J(x_*) - o(\lambda) \ge -o(\lambda). \end{split}$$

Thus, assumptions for multidirectional MVI are satisfied with  $\rho=-o(\lambda)$ 

Finding subgradients of the functional f we analyze relation between them and derive optimality conditions by taking limit as  $\lambda \downarrow 0$ .

Multidirectional Mean Value Inequalities are useful for developing calculus for nonsmooth functions

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- New more general variant of multidirectional MVI was presented
- Such MVI can be applied for derivation of optimality conditions for nonsmooth problems of Calculus of Variations

#### Robert Kipka and Kipka Junior



# THANK YOU