Multidirectional Mean Value Inequalities and Dynamic Optimization

Yuri Ledyaev

Department of Mathematics

Western Michigan University

and

Steklov Mathematical Institute

ledyaev@wmich.edu

Ulisse Dini (1845 – 1918)

Dini directional derivative
for $f : \mathbb{R}^n \to (-\infty, +\infty]$

$$\underline{D}\, f(x; v) := \liminf_{t\downarrow 0, w\to v} \frac{f(x+tw) - f(x)}{t}$$

Nonsmooth (non-differentiable) Functions are Natural

Envelopes of parametric families $\{f_\gamma(x)\}_{\gamma \in \Gamma}$

$$f(x) = \sup_{\gamma \in \Gamma} f_\gamma(x), \quad h(x) = \inf_{\gamma \in \Gamma} f_\gamma(x)$$

(sup-envelope and inf-envelope)

Nonsmooth (non-differentiable) Functions are Natural

Envelopes of parametric families $\{f_\gamma(x)\}_{\gamma\in\Gamma}$

$$f(x) = \sup_{\gamma\in\Gamma} f_\gamma(x), \quad h(x) = \inf_{\gamma\in\Gamma} f_\gamma(x)$$

(sup-envelope and inf-envelope)
k-th largest eigenvalue of a symmetric matrix $A \in \mathbb{R}^{n\times n}$:

$$\lambda_k(A) = \max_{X\in S(k,n)} \operatorname{tr}(X^T A X) - \max_{X\in S(k-1,n)} \operatorname{tr}(X^T A X)$$

where the Stiefel manifold

$$S(k,n) := \{X \in \mathbb{R}^{n\times k} : X^T X = I_k\}$$

Nonsmooth (non-differentiable) Functions are Natural

Nonsmooth control Lyapunov functions:

Finite dimensional nonlinear control system:

$$\dot{x}(t) \;=\; f(x(t), u(t)), \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{U}$$

x - state vector, u - control

(null-) asymptotic controllability:

for each x_0 $\exists$ "open loop" control $u : [0, +\infty) \to \mathbb{U}$

$$x(t; x_0, u) \to 0$$

feedback stabilizer:

$k : \mathbb{R}^n \to \mathbb{U}$ s.t. all trajectories of the system

$$\dot{x} \;=\; f(x, k(x))$$

$x(t) \to 0$ in some uniform and stable manner

Nonsmooth (non-differentiable) Functions are Natural

Long-standing question:

Asymptotic controllability ⇒ $\exists$ stabilizing feedback control?

Nonsmooth (non-differentiable) Functions are Natural

Long-standing question:

Asymptotic controllability ⇒ ∃ stabilizing feedback control?

Obvious answer if ∃ smooth control Lyapunov function V and continuous feedback $k(x)$ s.t.

$$\langle V'(x), f(x, k(x))\rangle \leq -W(x) < 0 \quad \forall\, x \neq 0$$

Nonsmooth (non-differentiable) Functions are Natural

Long-standing question:

Asymptotic controllability ⇨ $\exists$ stabilizing feedback control?

Obvious answer if $\exists$ smooth control Lyapunov function V and continuous feedback $k(x)$ s.t.

$$\langle V'(x), f(x, k(x)) \rangle \leq -W(x) < 0 \quad \forall\, x \neq 0$$

But there are topological obstacles to existence of smooth control Lyapunov function and continuous feedback.

Nonsmooth (non-differentiable) Functions are Natural

Long-standing question:

Asymptotic controllability ➡ $\exists$ stabilizing feedback control?

Obvious answer if $\exists$ smooth control Lyapunov function V and continuous feedback $k(x)$ s.t.

$$\langle V'(x), f(x, k(x))\rangle \leq -W(x) < 0 \quad \forall\, x \neq 0$$

But there topological obstacles to existence of smooth control Lyapunov function and continuous feedback.

Sontag 1984 $\exists$ continuous control Lyapunov function

$$\min_{v \in \text{co}\, f(x,U)} \underline{D}V(x; v) \leq -W(x) < 0 \quad \forall\, x \neq 0$$

Nonsmooth (non-differentiable) Functions are Natural

Long-standing question:

Asymptotic controllability ⇒ $\exists$ stabilizing feedback control?

Obvious answer if $\exists$ smooth control Lyapunov function V and continuous feedback $k(x)$ s.t.

$$\langle V'(x), f(x, k(x))\rangle \leq -W(x) < 0 \quad \forall\, x \neq 0$$

But there topological obstacles to existence of smooth control Lyapunov function and continuous feedback.

Sontag 1984 $\exists$ continuous control Lyapunov function

$$\min_{v \in \operatorname{co} f(x,U)} \underline{D}V(x; v) \leq -W(x) < 0 \quad \forall\, x \neq 0$$

Affirmative answer: **Clarke, Ledyaev, Sontag, Subbotin 1996**

Asymptotic controllability $\overset{\text{IFF}}{\Longleftrightarrow}$ $\exists$ stabilizing feedback control

Infinitesimal Properties of Nonsmooth Functions

How to characterize infinitesimal properties of non-differentiable (nonsmooth) functions

Infinitesimal Properties of Nonsmooth Functions

Dini directional derivative
for $f : \mathbb{R}^n \to (-\infty, +\infty]$

$$\underline{D}\, f(x; v) := \liminf_{t \downarrow 0, w \to v} \frac{f(x + tw) - f(x)}{t}$$

Subgradient $\zeta \in \partial f(x)$ definition:
$\exists$ smooth $g : \mathbb{R}^n \to \mathbb{R}$ such that $\zeta = g'(x)$, function $f(y) - g(y)$ attains local minimum at x

Infinitesimal Properties of Nonsmooth Functions

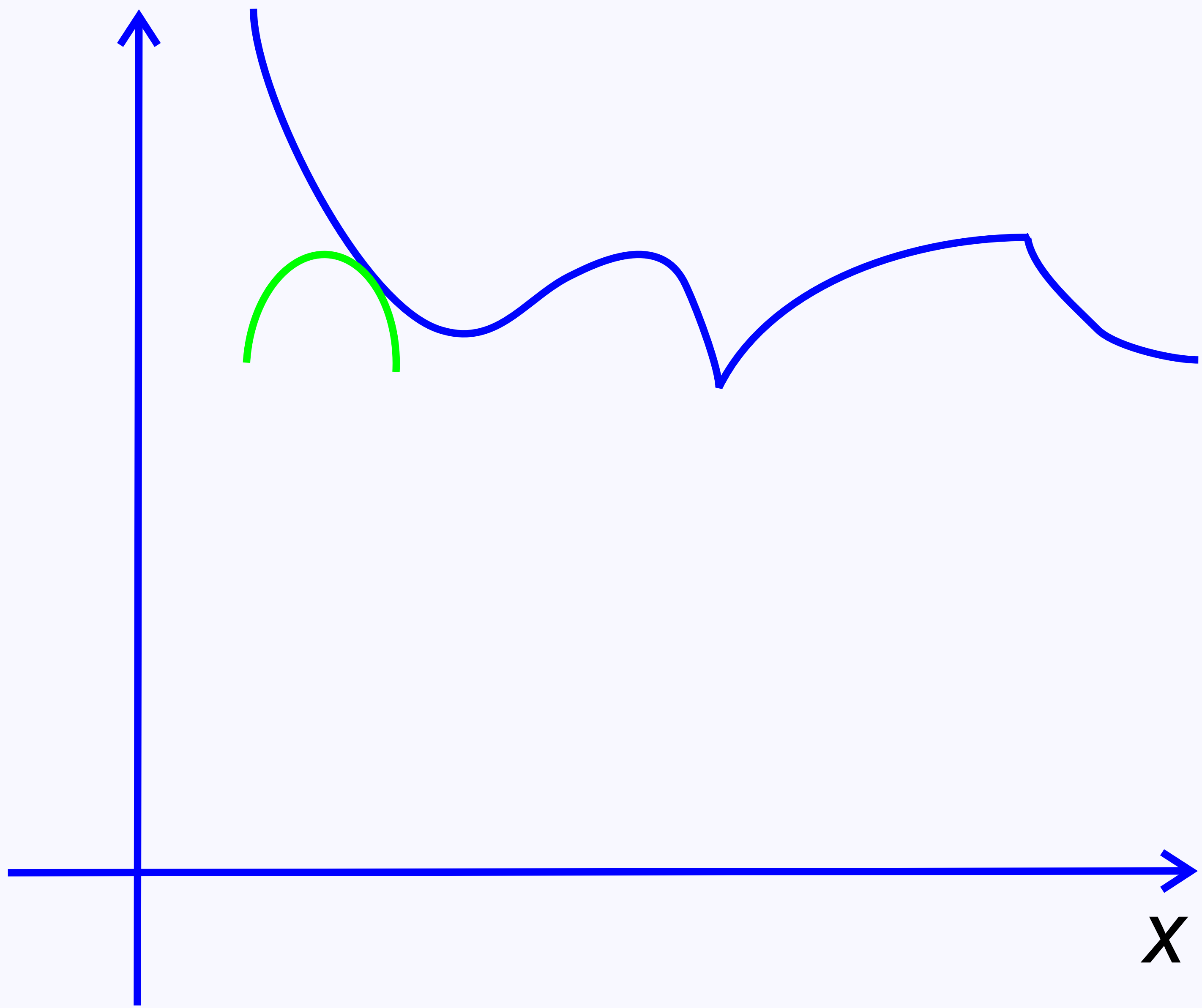
Dini directional derivative
for $f : \mathbb{R}^n \to (-\infty, +\infty]$

$$\underline{D}\, f(x; v) := \liminf_{t\downarrow 0, w\to v} \frac{f(x+tw) - f(x)}{t}$$

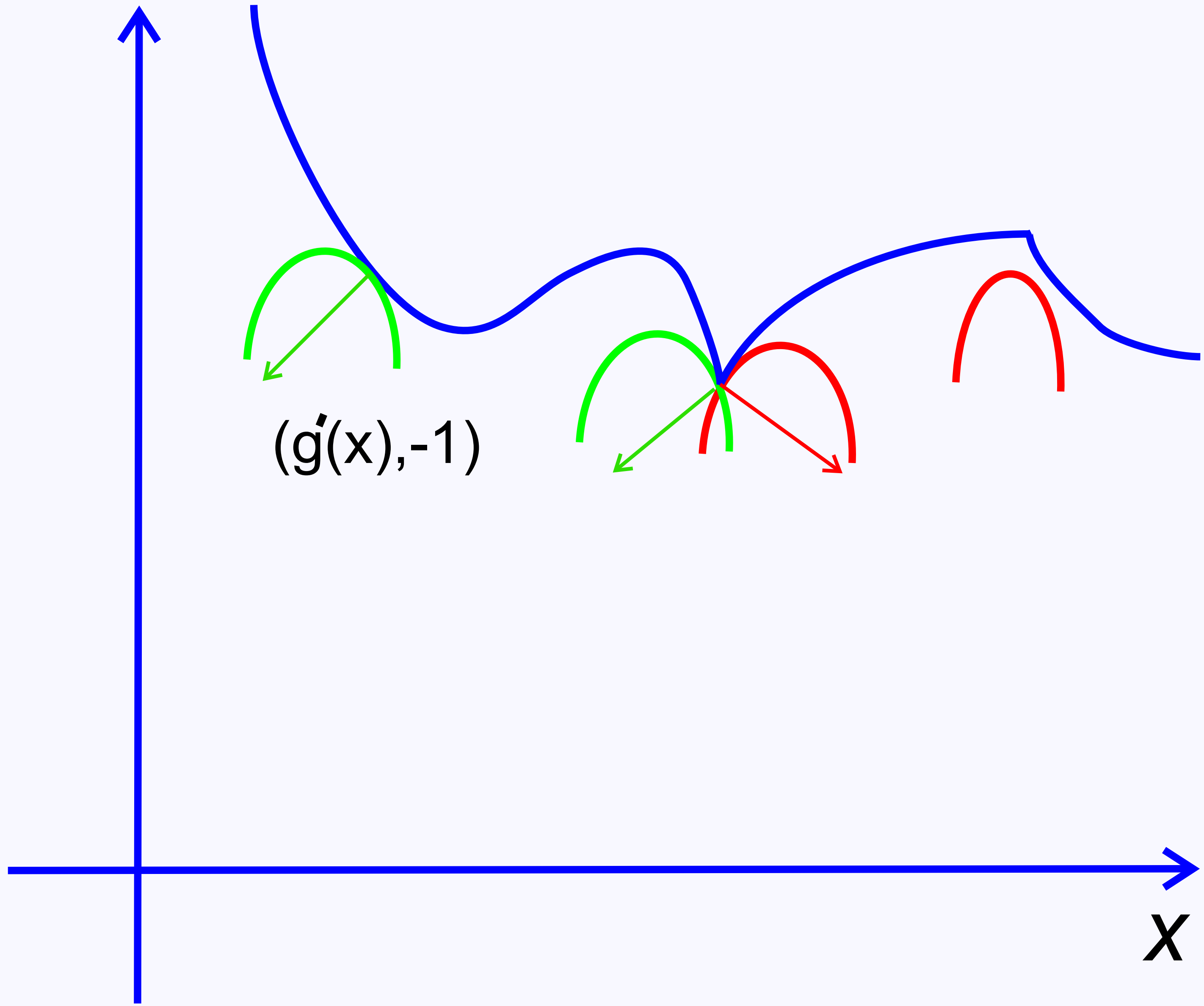
Subgradient $\zeta \in \partial f(x)$ definition:
$\exists$ smooth $g : \mathbb{R}^n \to \mathbb{R}$ such that $\zeta = g'(x)$, function $f(y) - g(y)$ attains local minimum at x

- $f(y)$ is convex, $g(y) = a + \langle \zeta, y \rangle$ is affine (subgradient in convex analysis sense)
- f is lower semicontinuous, $g(y) = a + \langle \zeta, y \rangle - \sigma \|y - x\|^2$ (proximal subgradient)
- f is lower semicontinuous, $g(y)$ is differentiable (smooth) ("Fréchet" subgradient)

Infinitesimal Properties of Nonsmooth Functions



Infinitesimal Properties of Nonsmooth Functions



Infinitesimal Properties of Nonsmooth Functions

Foundations of nonsmooth analysis:

Clarke, Danskin, Demyanov, Ioffe, Mordukhovich, Moreau, Pschenichny, Rockafellar, Rubinov, Subbotin, Vinter, Borwein

What is relation between subgradients and directional derivatives?

Infinitesimal Properties of Nonsmooth Functions

What is relation between subgradients and directional derivatives?

Obvious : if $\zeta \in \partial f(x)$ then (use $g(y) - g(x) \leq f(y) - f(x) \; \forall \; y$ near x)

$$\langle \zeta, v \rangle \leq \underline{D}f(x; v) \quad \forall v$$

Infinitesimal Properties of Nonsmooth Functions

What is a relation between subgradients and directional derivatives?

Obvious : if $\zeta \in \partial f(x)$ then

$$\langle \zeta, v \rangle \leq \underline{D} f(x; v) \quad \forall v$$

Non-obvious :

What is a relation between subgradients and directional derivatives?

Obvious : if $\zeta \in \partial f(x)$ then

$$\langle \zeta, v \rangle \leq \underline{D}f(x;v) \quad \forall v$$

Non-obvious : Subbotin's Theorem

THEOREM: Let $V \subset \mathbb{R}^n$ be nonempty, convex, and bounded, $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. For $\forall\ r$ such that

$$\inf_{v \in V} \underline{D}f(x;v) > r$$

and $\forall\ \varepsilon > 0\ \exists\ z \in x + \varepsilon B_{\mathbb{R}^n}$ with $|f(z) - f(x)| < \varepsilon$ and $\zeta \in \partial_F f(z)$ such that

$$\min_{v \in V} \langle \zeta, v \rangle > r$$

Infinitesimal Properties of Nonsmooth Functions

Applications of Subbotin's theorem:

- equivalence of Subbotin's minmax solutions (1980) of Hamilton-Jacobi equations and viscosity solutions
- for control Lyapunov function: condition

$$\min_{v \in \operatorname{co} f(x,U)} \underline{D}V(x;v) \le -W(x) < 0 \quad \forall\, x \neq 0$$

is equivalent to the condition

$$\min_{u \in U} \langle V', f(x,u) \rangle \le -W(x) < 0 \quad \forall\, V' \in \partial V(x), \quad x \neq 0$$

How to prove Subbotin's theorem?

Multidirectional Mean Value Inequalities

Classical (**Lagrange**) Mean value Theorem: $\exists\ z \in [x^0, x]$

$$f(x) - f(x^0) = \langle f'(z), x - x^0 \rangle$$

Multidirectional Mean Value Inequalities

Classical (**Lagrange**) Mean value Theorem: $\exists\ z \in [x^0, x]$

$$f(x) - f(x^0) = \langle f'(z), x - x^0 \rangle$$

Multidirectional Mean Value Inequality (**Clarke and Ledyaev**) for differentiable f, convex, closed and bounded set $X \subset \mathbb{R}^n$:
$\exists\ z \in [x^0, X] := \mathrm{co}\,(\{x^0\} \cup X)$

$$\min_{x \in X} f(x) - f(x^0) \leq \langle f'(z), x - x^0 \rangle \quad \forall\ x \in X$$

Multidirectional Mean Value Inequalities

Classical (**Lagrange**) Mean value Theorem: $\exists\ z \in [x^0, x]$

$$f(x) - f(x^0) = \langle f'(z), x - x^0 \rangle$$

Multidirectional Mean Value Inequality (**Clarke and Ledyaev**) for differentiable f, convex, closed and bounded set $X \subset \mathbb{R}^n$:
$\exists\ z \in [x^0, X] := \mathrm{co}\,(\{x^0\} \cup X)$

$$\min_{x \in X} f(x) - f(x^0) \leq \langle f'(z), x - x^0 \rangle \quad \forall\ x \in X$$

Proof by using **"clever function"**:

$$h(t, y) := f(x^0 + t(y - x^0)) - f(x^0) - rt, \quad y \in X,\ t \in [0, 1]$$

where

$$r := \min_{x \in X} f(x) - f(x^0)$$

Multidirectional Mean Value Inequalities

Classical (**Lagrange**) Mean value Theorem: $\exists\ z \in [x^0, x]$

$$f(x) - f(x^0) = \langle f'(z), x - x^0 \rangle$$

Multidirectional Mean Value Inequality (**Clarke and Ledyaev**) for differentiable f, convex, closed and bounded set $X \subset \mathbb{R}^n$:
$\exists\ z \in [x^0, X] := \mathrm{co}\,(\{x^0\} \cup X)$

$$\min_{x \in X} f(x) - f(x^0) \le \langle f'(z), x - x^0 \rangle \quad \forall\ x \in X$$

General case (**Clarke and Ledyaev 1994**): lower semicontinuous $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$, $\mathbb{E}$ - Hilbert space, $X \subset \mathbb{E}$ closed, convex and bounded

Multidirectional Mean Value Inequalities

General case (**Clarke and Ledyaev 1994**): lower semicontinuous $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$, $\mathbb{E}$ - Hilbert space, $X \subset \mathbb{E}$ closed, convex and bounded

THEOREM: For $\forall\, r$ and $\varepsilon > 0$

$$r < \sup_{\delta>0} \inf_{x\in X+\delta B} f(x) - f(x^0)$$

$\exists\, z \in [x^0, X] + \varepsilon B$ and $\zeta \in \partial_P f(z)$

$$r < \langle \zeta, x - x^0 \rangle \quad \forall\, x \in X$$

and $f(z) \leq \inf_{x\in X} f(x) + \max\{0, r\} + \varepsilon$.

Multidirectional Mean Value Inequalities

General case (**Clarke and Ledyaev 1994**): lower semicontinuous $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$, $\mathbb{E}$ - Hilbert space, $X \subset \mathbb{E}$ closed, convex and bounded

THEOREM: For $\forall\, r$ and $\varepsilon > 0$

$$r < \sup_{\delta > 0} \inf_{x \in X + \delta B} f(x) - f(x^0)$$

$\exists\, z \in [x^0, X] + \varepsilon B$ and $\zeta \in \partial_P f(z)$

$$r < \langle \zeta, x - x^0 \rangle \quad \forall\, x \in X$$

Generalizations: **Aussel, Corvellec, Lassonde**, **Clarke, Radulescu**, **Ivanov**, **Zlateva**, **Zhu et al.**

Multidirectional Mean Value Inequalities

Generalization of Subbotin's theorem for smooth Banach spaces $\mathbb{E}$:

THEOREM: Let $V \subset \mathbb{E}$ be nonempty, convex, and bounded, $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. For $\forall\, r$ such that

$$\inf_{v \in V} \underline{D}^w f(x; v) > r$$

and $\forall\, \varepsilon > 0\ \exists\, z \in x + \varepsilon B_{\mathbb{R}^n}$ with $|f(z) - f(x)| < \varepsilon$ and $\zeta \in \partial_F f(z)$ such that

$$\min_{v \in V} \langle \zeta, v \rangle > r$$

Multidirectional Mean Value Inequalities

Generalization of Subbotin's theorem for smooth Banach spaces $\mathbb{E}$:

THEOREM: Let $V \subset \mathbb{E}$ be nonempty, convex, and bounded, $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. For $\forall\, r$ such that

$$\inf_{v \in V} \underline{D}^w f(x; v) > r$$

and $\forall\, \varepsilon > 0\ \exists\, z \in x + \varepsilon B_{\mathbb{R}^n}$ with $|f(z) - f(x)| < \varepsilon$ and $\zeta \in \partial_F f(z)$ such that

$$\min_{v \in V} \langle \zeta, v \rangle > r$$

where

$$\underline{D}^w f(x; v) := \liminf_{t \downarrow 0,\, u \xrightarrow{w} v} \frac{f(x + tu) - f(x)}{t}$$

Implicit Multifunction Theorems

Let $F(x,p) = G(x,p) - K$ where $G : \mathbb{E} \times \mathbb{P} \to \mathbb{Y}$ smooth function, $K \subset \mathbb{Y}$ - closed convex cone.
Implicit multifunction

$$X(p) := \{x \in \mathbb{X} \ : \ G(x,p) \in K \text{ or } 0 \in F(x,p)\}$$

Let $0 \in F(x_0, p_0)$ and $G_x(x_0, p_0)\mathbb{X} - K = \mathbb{Y}$.
Then for $\forall\ x, p$ near x_0, p_0

$$X(p) \neq \emptyset, \qquad d(x, X(p) \leq kd(0, F(x,p))$$

Applications of Multidirectional MVI

Let $f : \mathbb{E} \to \mathbb{R}$ be lower semicontinuous

$$S := \{x \in \mathbb{E} \ : \ f(x) \leq 0\}$$

Assumptions: $\exists \ \Delta > 0$ such that $\forall \ x \notin S$

$$\|\zeta\| > \Delta \quad \forall \ \zeta \in \partial_F f(x)$$

If $S = \emptyset$ then for any $\rho > 0$ by the multidirectional MVI $\exists \ \zeta \in \partial_F f(z)$

$$\inf_{x \in x^0 + \rho B} f(x) \leq f(x^0) + \min_{x \in x^0 + \rho B} \langle \zeta, x - x^0 \rangle = f(x^0) - \rho\|\zeta\| < f(x^0) - \rho\Delta$$

Contradiction if $\rho > f(x^0)/\Delta$! Thus, $(x \in x^0 + \rho B) \cap S \neq \emptyset$ and

$$d_S(x^0) \leq \frac{f(x^0)}{\Delta}$$

METRIC REGULARITY and SOLVABILITY

Applications of Multidirectional MVI

Let $F(x,p)$ be multifunction , $F(x,p) \subset \mathbb{Y}$

Applications of Multidirectional MVI

Let $F(x,p)$ be multifunction , $F(x,p) \subset \mathbb{Y}$

Consider **implicit multifunction** $X(p)$

$$X(p) := \{x \in \mathbb{E} \ : \ 0 \in F(x,p)\}$$

define $f(x,p) := d_{F(x,p)}(0)$

and obtain

"Implicit Multifunction Theorems" (1999) Ledyaev and Zhu

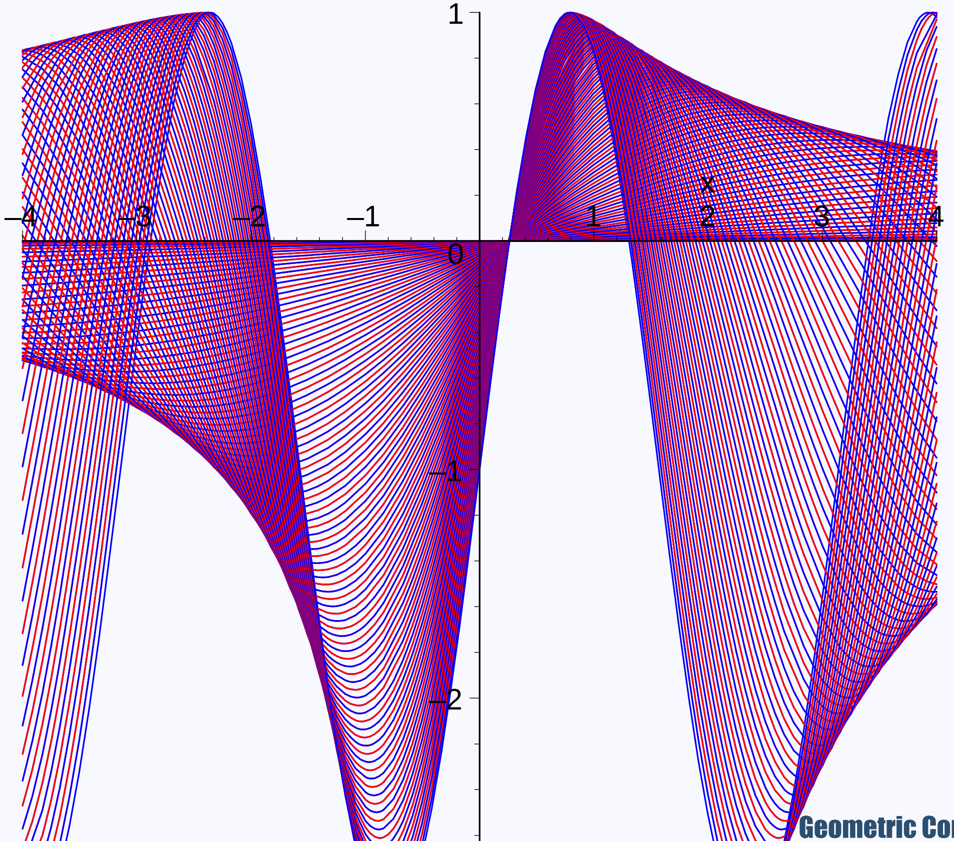
even earlier **'Theorems on Implicitly-Defined Multi-Valued Mappings" (1984) Ledyaev**

Applications of Multidirectional MVI

Representation of subgradients of sup- and inf-envelopes of parametric family $\{f_\gamma(x)\}_{\gamma\in\Gamma}$

$$f(x) := \sup_{\gamma\in\Gamma} f_\gamma(x), \quad h(x) := \inf_{\gamma\in\Gamma} f_\gamma(x)$$

$$f_\gamma(x) := \gamma \sin(\gamma x) - \frac{\gamma^2}{4}, \;\; \gamma \in [-2, 2]$$



Applications of Multidirectional MVI

Representation of subgradients of sup- and inf-envelopes of parametric family $\{f_\gamma(x)\}_{\gamma\in\Gamma}$

$$f(x) := \sup_{\gamma\in\Gamma} f_\gamma(x), \quad h(x) := \inf_{\gamma\in\Gamma} f_\gamma(x)$$

Let $\zeta \in \partial_F f(x)$. Then, $\forall\ \varepsilon > 0$ and $\forall \delta > 0$, $\exists$ convex coefficients $\{\alpha_k\}$, pairs $(x_k, \gamma_k) \in G_\delta(x)$, $k = 1, \ldots, K$, such that

$$\zeta \in \sum_{k=1}^{K} \alpha_k \partial_F f_{\gamma_k}(x_k) + \varepsilon B$$

If, in addition, the functions f_γ are continuous then the set $G_\delta(x)$ can be replaced by the smaller set $G^0_\delta(x)$.

$$G_\delta(x) := \{(z,\gamma) \in \mathbb{X}\times\Gamma : \ z \in [x,y]+\delta B\ \exists\, y \in x+\delta B, f_\gamma(y) \geq f(x)-\delta\}$$

Applications of Multidirectional MVI

Representation of subgradients of sup- and inf-envelopes of parametric family $\{f_\gamma(x)\}_{\gamma\in\Gamma}$

$$f(x) := \sup_{\gamma\in\Gamma} f_\gamma(x), \quad h(x) := \inf_{\gamma\in\Gamma} f_\gamma(x)$$

Let $\zeta \in \partial_F f(x)$. Then, $\forall\ \varepsilon > 0$ and $\forall \delta > 0$, $\exists$ convex coefficients $\{\alpha_k\}$, pairs $(x_k, \gamma_k) \in G_\delta(x)$, $k = 1, \ldots, K$, such that

$$\zeta \in \sum_{k=1}^{K} \alpha_k \partial_F f_{\gamma_k}(x_k) + \varepsilon B$$

If, in addition, the functions f_γ are continuous then the set $G_\delta(x)$ can be replaced by the smaller set $G^0_\delta(x)$.

$$G_\delta(x) := \{(z,\gamma) \in \mathbb{X}\times\Gamma : \ z \in [x,y]+\delta B\ \exists\, y \in x+\delta B, f_\gamma(y) \geq f(x)-\delta\}$$

In the case $x \in \mathbb{R}^n$ then $K = n + 1!$

Applications of Multidirectional MVI

Applications for manifolds of nonpositive curvature:

Jung theorem's generalization on radius of circumscribed ball

$$R(\mathbb{S}) \leq \frac{1}{2}\sqrt{\frac{2n}{n+1}}\; D(\mathbb{S})$$

Helly theorem generalization for intersection of convex sets on manifolds of nonpositive curvature

Ledyaev, Treiman and Zhu

THEOREM: Let $\mathcal{K} = \{K_\gamma\}_{\gamma\in\Gamma}$ be a family of convex sets and Γ is finite, or all K_γ are compact. Then if any $n+1$ sets from $\mathcal{K}$ have a common point, all sets K_γ have a common point.

Applications of Multidirectional MVI

Representation of subgradients of sup- and inf-envelopes of parametric family $\{f_\gamma(x)\}_{\gamma\in\Gamma}$

$$f(x) := \sup_{\gamma\in\Gamma} f_\gamma(x), \quad h(x) := \inf_{\gamma\in\Gamma} f_\gamma(x)$$

Let $\zeta \in \partial_F h$ ➡ $\exists$ function $\varphi(\varepsilon)$ s.t. $\varphi(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and $\forall\ \varepsilon > 0$, and $\forall\ (\gamma_\varepsilon, x_\varepsilon)$ with

$$x_\varepsilon \in x + \varepsilon B \ \text{ and } f_{\gamma_\varepsilon}(x_\varepsilon) < h(x) + \varepsilon$$

there exist $z \in x + 2\sqrt{\varepsilon}B$ s.t.

$$f_{\gamma_\varepsilon}(z) < h(x) + O(\sqrt{\varepsilon})$$

and

$$\zeta \in \partial_F f_{\gamma_\varepsilon}(z) + \varphi(\varepsilon)B.$$

Applications of Multidirectional MVI

Application:
Existence and uniqueness of minimizers!
For, example (for simplicity $\mathbb{E}$ is Hilbert):

$$h(x^\star) := \inf_{y \in S} \left(f(y) + \langle x^\star, y \rangle \right)$$

If for some (a.a.) $x^\star \in \mathbb{E}^\star$ $\exists \zeta \in \partial_F h(x^\star) \neq \emptyset$ ➡ $\forall \varepsilon > 0 \ \ \exists\, y_\varepsilon \in S$, $f(y_\varepsilon) + \langle x^\star, y_\varepsilon \rangle < h(x^\star) + \varepsilon$

$$\zeta \in y_\varepsilon + \varphi(\varepsilon) B$$

➡ ε-minimizer y_ε converges to unique minimizer $y_0 = \zeta$ of the function

$$y \to f(y) + \langle x^\star, y \rangle$$

on bounded S - generalization of Stegall variational principle

Applications of Multidirectional MVI

Uniqueness of closest points to the set $S \subset \mathbb{E}$,for simplicity $\mathbb{E}$ is Hilbert.
Consider function

$$h(x) := \inf_{y \in S} \|x - y\|^2 = d_S^2(x)$$

Let $\zeta \in \partial_F h(x)$ ➡ $\forall \varepsilon > 0 \; \exists \; y_\varepsilon \in S$

$$\|x - y_\varepsilon\|^2 < h(x) + \varepsilon, \quad \zeta = 2(x - y_\varepsilon) + \varphi(\varepsilon)B$$

➡ y_ε converges to $y_0 = x - \zeta/2$ unique closest to x point in S.
More results in

Ledyaev and Treiman "Sub- and Super-gradients of Envelopes, Semicontinuous Closures and Limits of Functions"

Nonsmooth Calculus of Variations Problems

Application of multidirectional MVI to deriving optimality conditions for nonsmooth calculus of variations problem

Nonsmooth Calculus of Variations Problems

Application of multidirectional MVI to deriving optimality conditions for nonsmooth calculus of variations problem

Consider minimizer x_* of a nonsmooth functional $J : W^{1,p} \to \mathbb{R} \cup \{+\infty\}$ of the form

$$J(x) = \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))\, dt,$$

where $W^{1,p}$ $(1 \leq p \leq \infty)$ denotes the Banach space of absolutely continuous maps $x : [a, b] \to \mathbb{R}^d$ satisfying $\dot{x} \in L^p(a, b; \mathbb{R}^n)$.

Nonsmooth Calculus of Variations Problems

Application of multidirectional MVI to deriving optimality conditions for nonsmooth calculus of variations problem

Consider minimizer x_* of a nonsmooth functional $J : W^{1,p} \to \mathbb{R} \cup \{+\infty\}$ of the form

$$J(x) = \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))\, dt,$$

where $W^{1,p}$ $(1 \le p \le \infty)$ denotes the Banach space of absolutely continuous maps $x : [a, b] \to \mathbb{R}^d$ satisfying $\dot{x} \in L^p(a, b; \mathbb{R}^n)$.

Motivation: Nonsmooth calculus of variations problems on manifolds. No exact proofs in the literature even for smooth problems!

Nonsmooth Calculus of Variations Problems

Application of multidirectional MVI to deriving optimality conditions for nonsmooth calculus of variations problem

Consider minimizer x_* of a nonsmooth functional $J : W^{1,p} \to \mathbb{R} \cup \{+\infty\}$ of the form

$$J(x) = \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))\, dt,$$

where $W^{1,p}$ $(1 \leq p \leq \infty)$ denotes the Banach space of absolutely continuous maps $x : [a, b] \to \mathbb{R}^d$ satisfying $\dot{x} \in L^p(a, b; \mathbb{R}^n)$.

Motivation: Nonsmooth calculus of variations problems on manifolds. No exact proofs in the literature even for smooth problems!

Difficulties for manifolds – combining global and local!

Nonsmooth Calculus of Variations Problems

Consider minimizer x_* of a nonsmooth functional $J : W^{1,p} \to \mathbb{R} \cup \{+\infty\}$ of the form

$$J(x) = \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))\, dt,$$

where $W^{1,p}$ $(1 \le p \le \infty)$ denotes the Banach space of absolutely continuous maps $x : [a, b] \to \mathbb{R}^d$ satisfying $\dot{x} \in L^p(a, b; \mathbb{R}^n)$.

Motivation: Nonsmooth calculus of variations problems on manifolds. No exact proofs in the literature even for smooth problems!

Difficulties for manifolds – combining global and local!

Kipka and Ledyaev "Optimal control of differential inclusions on manifolds" based on

Kipka and Ledyaev "Extension of Chronological Calculus for Dynamical Systems on Manifolds"

Nonsmooth Calculus of Variations Problems

Consider minimizer x_* of a nonsmooth functional $J : W^{1,p} \to \mathbb{R} \cup \{+\infty\}$ of the form

$$J(x) = \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))\, dt,$$

Long history: starting with **Clarke**, in 1990s important developments **Mordukhovich, Rockafellar, Ioffe, Smirnov, Vinter**

Nonsmooth Calculus of Variations Problems

Consider minimizer x_* of a nonsmooth functional $J : W^{1,p} \to \mathbb{R} \cup \{+\infty\}$ of the form

$$J(x) = \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))\, dt,$$

Long history: starting with **Clarke**, in 1990s important developments **Mordukhovich, Rockafellar, Ioffe, Smirnov, Vinter**

For problems with differential constraints

$$\dot{x}(t) \in F(t, x(t))$$

consider penalized nonsmooth functional with

$$L(t, x, \dot{x}) := L_0(t, x, \dot{x}) + kd(\dot{x}, F(t, x))$$

Nonsmooth Calculus of Variations Problems

Consider minimizer x_* of a nonsmooth functional $J : W^{1,p} \to \mathbb{R} \cup \{+\infty\}$ of the form

$$J(x) = \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))\, dt,$$

Long history: starting with **Clarke**, in 1990s important developments **Mordukhovich, Rockafellar, Ioffe, Smirnov, Vinter**

In this work we generalize a result by Clarke from
Clarke“ **Functional analysis, calculus of variations, and optimal control**”, Springer-Verlag, 2013.
Some assumptions are relaxed and shorter proof based on MVI

Assumptions and Optimality Conditions

Minimize

$$J(x) = \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))\, dt,$$

for $x \in W^{1,p}$ s.t. $\dot{x}(t) \in V(t)$ a.a. t
$V(t)$ measurable multifunction with closed convex values

Assumptions and Optimality Conditions

Minimize

$$J(x) = \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))\, dt,$$

for $x \in W^{1,p}$ s.t. $\dot{x}(t) \in V(t)$ a.a. t
$V(t)$ measurable multifunction with closed convex values

ASSUMPTIONS:

- (Controllability) $\exists\ \delta_* > 0$ s.t. for a.a. t, $\dot{x}_*(t) + \delta_* B \subset V(t)$
- Function $\ell : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous;
- $\forall\ (x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ the function $t \mapsto L(t, x, v)$ is Lebesgue measurable;
- For a given $x_* \in W^{1,p}$ there exists $\varepsilon_* > 0$ such that for some $k_L \in L^2(a, b; \mathbb{R})$, for a.a. $t \in [a, b]$, $\forall\ x, y \in x_*(t) + \varepsilon B$, and $\forall$ $u, v \in V(t)$

$$|L(t, x, u) - L(t, y, v)| \leq k_L(t) \{\|x - y\|_{\mathbb{R}^d} + \|u - v\|_{\mathbb{R}^d}\}$$

Assumptions and Optimality Conditions

Type of minimizer:
x_* is a V-local minimizer for J if $\exists\ \varepsilon_* > 0$ s.t.$\forall\ x \in W^{1,p}$ satisfying $\dot{x}(t) \in V(t)$ for a.a. $t \in [a,b]$ and $\|x - x_*\|_\infty < \varepsilon_*$ ⇒ $J(x) \geq J(x_*)$.

Assumptions and Optimality Conditions

THEOREM: Let x_* be V-local minimizer under assumptions. Then $\exists$ absolutely continuous $p : [a, b] \to \mathbb{R}^d$ which satisfies the Euler inclusion:

$$\dot{p}(t) \in \mathrm{co}\,\{w\ :\ (w, p(t)) \in \partial_L L(t, x_*(t), \dot{x}_*(t))\} \quad a.a.\ t \in [a, b]\,,$$

the transversality conditions:

$$(p(a), -p(b)) \in \partial_L \ell(x_*(a), x_*(b))$$

Weierstrass condition: a.a. $t \in [a, b]$, $\forall\ v \in V(t)$,

$$L(t, x_*(t), v) \geq L(t, x_*(t), \dot{x}_*(t)) + \langle p(t), v - \dot{x}_*(t) \rangle\,.$$

Assumptions and Optimality Conditions

THEOREM: Let x_* be V-local minimizer under assumptions. Then $\exists$ absolutely continuous $p : [a,b] \to \mathbb{R}^d$ which satisfies the Euler inclusion:

$$\dot{p}(t) \in \mathrm{co}\,\{w\ :\ (w, p(t)) \in \partial_L L(t, x_*(t), \dot{x}_*(t))\} \quad a.a.\ t \in [a,b]\,,$$

the transversality conditions:

$$(p(a), -p(b)) \in \partial_L \ell(x_*(a), x_*(b))$$

Weierstrass condition: a.a. $t \in [a,b]$, $\forall\ v \in V(t)$,

$$L(t, x_*(t), v) \geq L(t, x_*(t), \dot{x}_*(t)) + \langle p(t), v - \dot{x}_*(t) \rangle\,.$$

Smooth case $p(t) = L_v(t, x_*(t), \dot{x}_*(t)),\ p'(t) = L_x(t, x_*(t), \dot{x}_*(t))$

New Multidirectional MVI

Consider linear mappings $A : \mathcal{V} \to \mathbb{E}$ and $\psi : \mathcal{V} \to \mathbb{R}$, where $\mathcal{V}$ is an arbitrary convex set.
In particular, we make the following assumptions on the data $f, X, \mathcal{V}, A$ and ψ:

A1: The function $f : \mathbb{E} \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous, $f(x_0 + A(v_0))$ is finite, and f is bounded from below on the set

$$D_0 := [x_0 + A(v_0), X + A(\mathcal{V})] + \delta B,$$

for some $\delta > 0$;

A2: The set $X \subset \mathbb{E}$ is closed, bounded, and convex;

A3: The sets $A(\mathcal{V})$ and $\psi(\mathcal{V}) \subset \mathbb{R}$ are bounded.

New Multidirectional MVI

Kipka and Ledyaev

THEOREM: Let $\mathbb{E}$ admits a bump function b(globally Lipschitz and β-smooth), Assumptions **A1 – A3** hold.
Then $\forall\ \varepsilon > 0$ and $\forall\ \rho$ s.t.

$$\rho < \sup_{\delta>0} \inf_{x\in X+\delta B,\ v\in\mathcal{V}} \left\{f(x+A(v)) - f(x^0+A(v^0)) + \psi(v) - \psi(v^0)\right\}$$

$\exists\ z \in \left[x^0 + A(v^0), X + A(\mathcal{V})\right] + \varepsilon B$ and $z^* \in \partial_\beta f(z)$ s.t.

$$\rho < \left\langle z^*, x - x^0 + A(v) - A(v^0)\right\rangle + \psi(v) - \psi(v^0)$$

$\forall\ x \in X$ and $v \in \mathcal{V}$.

More general than existing variants of MVI.

New Multidirectional MVI

Proof is based on the use of bump function instead of square of Hilbert norm in **Clarke and Ledyaev 1994**

Optimality Conditions for Nonsmooth Calculus of Variations

Traditional variation of minimizer:

$$x^{\lambda}(t) = x_*(t) + \lambda y(t), \quad \dot{y}(t) = v(t)$$

Optimality Conditions for Nonsmooth Calculus of Variations

Use relaxed controls $\mu(t)$ instead $v(t)$

$$\mu(t) = \sum_{i=1}^{m} \alpha_i \delta_{u_i(t)}$$

$\alpha_i \geq 0$ satisfy $\sum_{i=1}^{m} \alpha_i = 1$

Optimality Conditions for Nonsmooth Calculus of Variations

Use relaxed controls $\mu(t)$ instead $v(t)$

$$\mu(t) = \sum_{i=1}^{m} \alpha_i \delta_{u_i(t)}$$

$\alpha_i \geq 0$ satisfy $\sum_{i=1}^{m} \alpha_i = 1$
Then relaxed variation

$$x^\lambda(t) = x_*(t) + \lambda v + \lambda \int_a^t w(s)\, ds + \int_a^t \int_{\mathbb{R}^d} u\, \mu^\lambda(s; du)\, ds$$

$$J^\lambda := \ell(x^\lambda(a), x^\lambda(b)) + \int_a^b L(t, x^\lambda(t), \dot{x}_*(t) + \lambda w(t) + u)\mu^\lambda(t; du)\, dt$$

where μ is a relaxed velocity, $\mu^\lambda := (1-\lambda)\delta_{u_*(t)} + \lambda\mu$, $v \in \mathbb{R}^d$, and $w \in L^\infty(a, b; \mathbb{R}^d)$, $u_*(t) = \dot{x}_*(t)$

Optimality Conditions for Nonsmooth Calculus of Variations

Then relaxed variation

$$x^\lambda(t) = x_*(t) + \lambda v + \lambda \int_a^t w(s)\,ds + \int_a^t \int_{\mathbb{R}^d} u\,\mu^\lambda(s;du)\,ds$$

$$J^\lambda := \ell(x^\lambda(a), x^\lambda(b)) + \int_a^b L(t, x^\lambda(t), \dot{x}_*(t) + \lambda w(t) + u)\mu^\lambda(t;du)\,dt$$

where μ is a relaxed velocity, $\mu^\lambda := (1-\lambda)\delta_{u_*(t)} + \lambda\mu$, $v \in \mathbb{R}^d$, and $w \in L^\infty(a,b;\mathbb{R}^d)$, $u_*(t) = \dot{x}_*(t)$

Approximation: Any $(x^\lambda, J^\lambda$ can be approximated with arbitrary precision by traditional control $u(t)$! Controllability is used here for lower semicontinuous ℓ.

Optimality Conditions for Nonsmooth Calculus of Variations

Let $\mathbb{E}$ denote the Banach space $\mathbb{R}^d \times \mathbb{R}^d \times L^2(a,b;\mathbb{R}^d) \times L^2(a,b;\mathbb{R}^d)$, define a function $f : E \to \mathbb{R} \cup \{+\infty\}$ through

$$f(x,y,u,v) = \ell(x_*(a)+x, x_*(b)+y) + \int_a^b \widetilde{L}(t, x_*(t)+u(t), \dot{x}_*(t)+v(t))\,dt.$$

Define linear maps $A : \mathcal{V}(R,\varepsilon) \to \mathbb{E}$ and $\psi : \mathcal{V}(R,\varepsilon) \to$ through

$$A(\mu) = \left(0, \int_a^b \int_{\mathbb{R}^d} u\mu(s,du)\,ds,\ \int_a^{(\cdot)} \int_{\mathbb{R}^d} u\mu(s,du)\,ds, 0\right)$$

$$\psi(\mu) = \int_a^b \int_{\mathbb{R}^d} \widetilde{L}(s, x_*(s), \dot{x}_*(s)+u)\mu(s,du)\,ds$$

Then

$$f(x+A(\mu)) - f(0+A(\delta_0)) + \psi(\mu) - \psi(\delta_0) =$$

$$\ell(x^\lambda(a), x^\lambda(b)) - \ell(x_*(a), x_*(b)) + \int_a^b \widetilde{L}(t, x^\lambda(t), \dot{x}_*(t) + +\lambda w(t))\, dt +$$

$$+ \lambda \int_a^b \int_{\mathbb{R}^d} \widetilde{L}(t, x_*(t), \dot{x}_*(t) + u)\, \mu(t, du) - \int_a^b \widetilde{L}(t, x_*(t), \dot{x}_*(t))\, dt =$$

$$= \ell(x^\lambda(a), x^\lambda(b)) + \int_a^b \int_{\mathbb{R}^d} \widetilde{L}(t, x^\lambda(t), \dot{x}_*(t) + \lambda w(t) + u) \mu^\lambda(t, du)\, dt -$$

$$- J(x_*) - o(\lambda) = J^\lambda - J(x_*) - o(\lambda) \geq -o(\lambda).$$

Then

$$f(x+A(\mu)) - f(0+A(\delta_0)) + \psi(\mu) - \psi(\delta_0) =$$

$$\ell(x^\lambda(a), x^\lambda(b)) - \ell(x_*(a), x_*(b)) + \int_a^b \widetilde{L}(t, x^\lambda(t), \dot{x}_*(t) + +\lambda w(t))\, dt +$$

$$+\lambda \int_a^b \int_{\mathbb{R}^d} \widetilde{L}(t, x_*(t), \dot{x}_*(t) + u)\, \mu(t, du) - \int_a^b \widetilde{L}(t, x_*(t), \dot{x}_*(t))\, dt =$$

$$= \ell(x^\lambda(a), x^\lambda(b)) + \int_a^b \int_{\mathbb{R}^d} \widetilde{L}(t, x^\lambda(t), \dot{x}_*(t) + \lambda w(t) + u) \mu^\lambda(t, du)\, dt -$$

$$- J(x_*) - o(\lambda) = J^\lambda - J(x_*) - o(\lambda) \geq -o(\lambda).$$

Thus, assumptions for multidirectional MVI are satisfied with $\rho = -o(\lambda)$

Optimality Conditions for Nonsmooth Calculus of Variations

Finding subgradients of the functional f we analyze relation between them and derive optimality conditions by taking limit as $\lambda \downarrow 0$.

Conclusions

- Multidirectional Mean Value Inequalities are useful for developing calculus for nonsmooth functions

Conclusions

- Multidirectional Mean Value Inequalities are useful for developing calculus for nonsmooth functions
- New more general variant of multidirectional MVI was presented

Conclusions

- Multidirectional Mean value Inequalities are useful for developing calculus for nonsmooth functions
- New more general variant of multidirectional MVI was presented
- Such MVI can be applied for derivation of optimality conditions for nonsmooth problems of Calculus of Variations

Conclusions

Robert Kipka and Kipka Junior

Conclusions

THANK YOU