Pontryagin maximum principle for the deterministic mean field type optimal control problem via the Lagrangian approach

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Mean field type control problem. Dynamics

Let

- $\mathbb{R}^d$ be a phase space for each agent;
- $f(t, x, m, u)$, where $t \in [0, T]$, $x \in \mathbb{R}^d$, m is a probability on $\mathbb{R}^d$, $u \in U$ is a control, be a nonlocal velocity field.

Dynamics of distribution of agents satisfies in the distributional sense the nonlocal continuity equation:

$$\frac{\partial}{\partial t} m(t) + \mathrm{div}(f(t, x, m(t), u(t, x))m(t)) = 0.$$

In particular, the dynamics of each agent obeys the ODE:

$$\dot{x} = f(t, x, m(t), u(t, x)).$$

Mean field type control problem.

- The agents play cooperatively to minimize the averaged payoff.
- The payoff of each agent is equal to

$$\sigma(x(T), m(T)) + \int_0^T f_0(t, x(t), m(t), u(t, x(t)))dt.$$

Notation

- If (X, ρ_X) is a Polish space, then $\mathcal{B}(X)$ denotes the Borel σ-algebra on X.
- $\mathcal{P}(X)$ is the set of Borel probabilities on X.

Push-forward measure

Assume that

- $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ are measurable spaces,
- $\mathbb{P}$ is a probability on $\mathcal{F}$,
- $\xi : \Omega \to \Omega'$ is measurable function.

A probability $\xi\sharp\mathbb{P}$ on $\mathcal{F}'$ defined by the rule: for $E \in \mathcal{F}'$,

$$(\xi\sharp\mathbb{P})(E) \triangleq \mathbb{P}(\xi^{-1}(E))$$

is called a push-forward measure.

Notation. Space of probabilities

- If (X, ρ_X) is a Polish space, $p \geq 1$, then $\mathcal{P}^p(X)$ is the set of probabilities on X with the finite p-th moment, i.e., $m \in \mathcal{P}^p(X)$ iff, for some (equivalently, any) $x_* \in X$,
$$\mathcal{M}_p^p(m) \triangleq \int_X \rho_X^p(x, x_*) m(dx) < \infty.$$
- Distance on $\mathcal{P}^p(X)$: if $m_1, m_2 \in \mathcal{P}^p(X)$, then
$$W_p(m_1, m_2) \triangleq \inf\Big[\int_{X\times X} \rho_X^p(x_1,x_2)\pi(dx_1 dx_2) : \\ \pi \in \Pi(m_1, m_2)\Big]^{1/p},$$
where $\Pi(m_1, m_2)$ is the set of probabilities π on $X \times X$ such that, for any measurable $E \subset X$, $\pi(E \times X) = m_1(X)$, $\pi(X \times E) = m_2(E)$.

Notation. State and controls

- Space of curves: $\Gamma = C([0,T];\mathbb{R}^d)$.
- Space of curves in the costate space: $\Gamma^* \triangleq C([0,T];\mathbb{R}^{d,*})$.
- Evaluation operator: for $\gamma \in \Gamma$,

$$e_t(\gamma) = \gamma(t).$$

- Space of controls: $\mathcal{U}^p \triangleq L^p([0,T],\mathcal{B}([0,T]),\lambda;U)$, where λ stands for the Lebesgue measure.

Lagrangian approach

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard probability space.
- Control process: (X, u_L), where $X \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \Gamma)$, $u_L \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{U}^p)$.
- Dynamics:

$$\frac{d}{dt}X(t,\omega) = f(t, X(t,\omega), X(t)\sharp\mathbb{P}, u_L(t,\omega)).$$

- Initial condition: $X(0)\sharp\mathbb{P} = m_0$.
- Payoff:

$$J_L(X, u_L) \triangleq \int_\Omega \sigma(X(T,\omega), X(T)\sharp\mathbb{P})\mathbb{P}(d\omega) + \int_\Omega \int_0^T f_0(t, X(t,\omega), X(t)\sharp\mathbb{P}, u_L(t,\omega)) dt \mathbb{P}(d\omega).$$

Kantorovich approach

- Control process: (η, u_K), where $\eta \in \mathcal{P}^p(\Gamma)$, $u_K \in L^p(\Gamma, \mathcal{B}(\Gamma), \eta; \mathcal{U}^p)$.
- Feasibility: for η-a.e. $\gamma \in \Gamma$,

$$\frac{d}{dt}\gamma(t) = f(t, \gamma(t), e_t\sharp\eta, u_K(t, \gamma)),$$

  where $e_t(\gamma) = \gamma(t)$, $(e_t\sharp\eta)(E) = \eta\{\gamma \in \Gamma : \gamma(t) \in E\}$.
- Initial condition: $e_0\sharp\eta = m_0$.
- Payoff:

$$J_K(\eta, u_K) \triangleq \int_\Gamma \sigma(\gamma(T), e_T\sharp\eta)\eta(d\gamma) + \int_\Gamma \int_0^T f_0(t, \gamma(t), e_t\sharp\eta, u_K(t, \gamma))dt\eta(d\gamma).$$

Eulerian approach

- Control process: $(m(\cdot), u_E)$, where $m(t)$ is a probability on $\mathbb{R}^d$, $u_E : [0, T] \times \mathbb{R}^d \to U$.
- Dynamics: $m(\cdot)$ is a distributional solution of the nonlocal continuity equation:

$$\partial_t m(t) + \operatorname{div}(v_E(t,x)m(t)) = 0,$$

  for $v_E(t,x) = f(t,x,m(t),u_E(t,x))$.
- Initial condition: $m(0) = m_0$.
- Payoff:

$$J_E(\mu, u_E) \triangleq \int_{\mathbb{R}^d} \sigma(x, m(T))m(T, dx) + \int_0^T \int_{\mathbb{R}^d} f_0(t, x, m(t), u_E(t,x))m(t, dx)dt.$$

Continuity equation

$m(\cdot)$ is a distributional solution of the nonlocal continuity equation:

$$\partial_t m(t) + \operatorname{div}(v(t,x)m(t)) = 0,$$

iff, for every $\varphi \in C_c^\infty((0,T) \times \mathbb{R}^d)$,

$$\int_0^T \int_{\mathbb{R}^d} \left[\partial_t \varphi(t,x) + \nabla_x \varphi(t,x) v(t,x)\right] m(t,x) dt = 0.$$

Intrinsic derivative

Definition

Let $\Phi : \mathcal{P}^p(\mathbb{R}^d) \to \mathbb{R}$. A function $\frac{\delta \Phi}{\delta m} : \mathcal{P}^p(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is a flat derivative iff, for any $m' \in \mathcal{P}^p(\mathbb{R}^d)$,

$$\lim_{s \downarrow 0} \frac{\Phi((1-s)m + sm') - \Phi(m)}{s} = \int_{\mathbb{R}^d} \frac{\delta \Phi}{\delta m}(m, y)[m'(dy) - m(dy)].$$

Intrinsic derivative

Definition

Let $\Phi : \mathcal{P}^p(\mathbb{R}^d) \to \mathbb{R}$. A function $\frac{\delta \Phi}{\delta m} : \mathcal{P}^p(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is a flat derivative iff, for any $m' \in \mathcal{P}^p(\mathbb{R}^d)$,

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Definition

The function $\nabla_m \Phi$ defined by the rule

$$\nabla_m \Phi(m, y) \triangleq \nabla_y \frac{\delta \Phi}{\delta m}(m, y)$$

is called an intrinsic derivative of the function Φ.

Assumptions

- $p > 1$;
- U is a closed subset of a separable Banach space;
- the functions f, f_0 and σ are continuous;
- there exists a constant C_∞ such that

$$\|f(t,x,m,u)\| \leq C_\infty(1 + \|x\| + \mathcal{M}_p(m) + \|u\|),$$

$$|f_0(t,x,m,u)| \leq C_\infty(1 + \|x\|^p + \mathcal{M}_p^p(m) + \|u\|^p),$$

$$|\sigma(x,m)| \leq C_\infty(1 + \|x\|^p + \mathcal{M}_p^p(m));$$

- the functions f and f_0 are continuous w.r.t. time variable with the same modulus of continuity $\varsigma(\cdot)$;
- the function f is continuously differentiable w.r.t. x and m; its derivatives $\nabla_x f$ and $\nabla_m f$ are bounded by constants C_x and C_m respectively;

Assumptions

- the function f_0 is continuously differentiable w.r.t. x and m; the derivatives $\nabla_x f_0$ and $\nabla_m f_0$ satisfy the following growth conditions with constants C_x^0, C_m^0:

$$\|\nabla_x f_0(t, x, m, u)\|^q \leq C_x^0(1 + \|x\|^p + \mathcal{M}_p^p(m) + \|u\|^p),$$

$$\|\nabla_m f_0(t, x, m, y, u)\|^q \leq C_m^0(1+\|x\|^p+\|y\|^p+\mathcal{M}_p^p(m)+\|u\|^p);$$

- the terminal payoff σ is continuously differentiable; the functions $\nabla_x\sigma$ and $\nabla_m\sigma$ satisfy the following estimates with some nonnegative constants C_x^σ, C_m^σ:

$$\|\nabla_x\sigma(x, m)\|^q \leq C_x^\sigma(1 + \|x\|^p + \mathcal{M}_p^p(m)),$$

$$\|\nabla_m\sigma(x, m, y)\|^q \leq C_m^\sigma(1 + \|x\|^p + \|y\|^p + \mathcal{M}_p^p(m)).$$

Lagrangian strong local L^p-minimizer

A Lagrangian control process (X^*, u_L^*) is a strong local L^p-minimizer at m_0 if there exists $\varepsilon > 0$ satisfying the following condition: for any admissible (X, u_L) such that $\|X - X^*\|_{L^p} \leq \varepsilon$,

$$J_L(X^*, u_L^*) \leq J_L(X, u_L).$$

Lagrangian strong local W_p-minimizer

A Lagrangian control process (X^*, u_L^*) is a strong local W_p-minimizer at m_0 if there exists $\varepsilon > 0$ satisfying the following condition: for every admissible (X, u_L) such that $W_p(X\sharp\mathbb{P}, X^*\sharp\mathbb{P}) \leq \varepsilon$,

$$J_L(X^*, u_L^*) \leq J_L(X, u_L).$$

Pontryagin local L^p-minimizer for the Lagrangian approach

A Lagrangian control process (X^*, u_L^*) is a Pontryagin local L^p-minimizer at m_0 if there exists $\varepsilon > 0$ such that inequality

$$J_L(X^*, u_L^*) \leq J_L(X, u_L).$$

holds true for each admissible (X, u_L) satisfying

- $\|X - X^*\|_{L^p} \leq \varepsilon$,
- $(\lambda \otimes \mathbb{P})\{(t, \omega) \in [0, T] \times \Omega : u^*(t, \omega) \neq u(t, \omega)\} \leq \varepsilon$.

Pontryagin local W_p-minimizer for the Lagrangian approach

A Lagrangian control process (X^*, u_L^*) is a Pontryagin local W_p-minimizer at m_0 if there exists $\varepsilon > 0$ such that

$$J_L(X^*, u_L^*) \leq J_L(X, u_L).$$

holds true for each admissible (X, u_L) satisfying

- $W_p(X\sharp\mathbb{P}, X^*\sharp\mathbb{P}) \leq \varepsilon$,
- $(\lambda \otimes \mathbb{P})\{(t,\omega) \in [0, T] \times \Omega : u^*(t,\omega) \neq u(t,\omega)\} \leq \varepsilon$.

Local minima within Lagrangian approach

- Every strong minimizer is a Pontryagin one.
- Every W_p minimizer is a L^p-minimizer.

The Pontryagin L^p-minimizer is the weakest minimizer!

Pontryagin maximum principle for the Lagrangian approach

Regular points

A point $t \in [0, T]$ is called regular for the process (X, u) if

- $\|u(t)\|_{L^p} < \infty$,
- $$\lim_{h\downarrow 0} \mathbb{E}\left\| \frac{1}{h}\int_t^{t+h} f(\tau, X(\tau), X(\tau)\sharp\mathbb{P}, u(\tau))d\tau - f(t, X(t), X(t)\sharp\mathbb{P}, u(t))\right\|^p = 0,$$
- $$\lim_{h\downarrow 0} \mathbb{E}\left| \frac{1}{h}\int_t^{t+h} f_0(\tau, X(\tau), X(\tau)\sharp\mathbb{P}, u(\tau))d\tau - f_0(t, X(t), X(t)\sharp\mathbb{P}, u(t))\right| = 0.$$

Regular points

Almost every $t \in [0, T]$ are regular.

Pontryagin function for the Lagrangian PMP

- local Pontryagin function: for $t \in [0, T]$, $x \in \mathbb{R}^d$, $m \in \mathcal{P}^p(\mathbb{R}^d)$, $\psi \in \mathbb{R}^{d,*}$, $u \in U$,

$$H(t, x, m, \psi, u) \triangleq \psi f(t, x, m, u) - f_0(t, x, m, u);$$

- global Pontryagin function: for $t \in [0, T]$, $X \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, $\Psi \in L^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{d,*})$, $u \in L^p(\Omega, \mathcal{F}, \mathbb{P}; U)$,

$$\mathbb{H}(t, X, \Psi, u) \triangleq \mathbb{E} H(t, X, X \sharp \mathbb{P}, \Psi, u).$$

Pontryagin maximum principle in the Lagrangian form

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space, a Lagrangian process (X^*, u^*) be a Pontryagin local L^p-minimizer. Then there exists a function $\Psi \in L^q(\Omega, \mathcal{F}, \mathbb{P}; \Gamma^*)$ such that

- the costate equation,
- the transversality condition,
- the maximization conditions

hold true.

Costate equation

For $\mathbb{P}$-a.e. $\omega \in \Omega$, $\Psi(\cdot, \omega)$ solves

$$\begin{aligned}
&\frac{d}{dt}\Psi(t,\omega) \\
&\quad = -\Psi(t,\omega)\nabla_x f(t, X^*(t,\omega), X^*(t)\sharp\mathbb{P}, u^*(t,\omega)) \\
&\qquad\quad + \nabla_x f_0(t, X^*(t,\omega), X^*(t)\sharp\mathbb{P}, u^*(t)) \\
&\qquad\quad - \int_\Omega \Psi(t,\omega')\nabla_m f(t, X^*(t,\omega'), X^*(t)\sharp\mathbb{P}, \\
&\qquad\qquad\qquad\qquad X^*(t,\omega), u^*(t,\omega'))\mathbb{P}(d\omega') \\
&\qquad\quad + \int_\Omega \nabla_m f_0(t, X^*(t,\omega'), X^*(t)\sharp\mathbb{P}, \\
&\qquad\qquad\qquad\qquad X^*(t,\omega), u^*(t,\omega'))\mathbb{P}(d\omega').
\end{aligned}$$

Transversality condition

$$\Psi(T,\omega)$$
$$= -\nabla_x \sigma(X^*(T,\omega), X^*(T)\sharp\mathbb{P})$$
$$- \int_\Omega \nabla_m \sigma(X^*(T,\omega'), X^*(T)\sharp\mathbb{P}, X^*(T,\omega))\mathbb{P}(d\omega')$$

$\mathbb{P}$-a.s.

Maximization of the Hamiltonian condition

- global form: for each regular point $t \in [0, T]$,

$$\mathbb{H}(s, X^*(s), \Psi(s), u^*(s)) = \max_{\nu \in L^p(\Omega, \mathcal{F}, \mathbb{P}; U)} \mathbb{H}(s, X^*(s), \Psi(s), \nu)$$

- local form: for each regular point $t \in [0, T]$,

$$\begin{aligned} H(s, X^*(s), & X^*(s)\sharp\mathbb{P}, \Psi(s), u^*(s)) \\ &= \max_{u \in U} H(s, X^*(s), X^*(s)\sharp\mathbb{P}, \Psi(s), u) \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Hamiltonian form of the costate equation

$$\begin{aligned}
&\frac{d}{dt}X^*(t) = \nabla_\Psi \mathbb{H}(t, X^*(t), \Psi(t), u^*(t)),\\
&\frac{d}{dt}\Psi(t) = -\nabla_X \mathbb{H}(t, X^*(t), \Psi(t), u^*(t)),\\
&X(0) = X_0,\\
&\Psi(T) = -\nabla_X \Sigma(X^*(T)).
\end{aligned}$$

Here,

- $\nabla_X \mathbb{H}$, $\nabla_\Psi \mathbb{H}$ stands for the derivatives w.r.t. $X \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ and $\Psi \in L^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$,
- $\Sigma(X) \triangleq \mathbb{E}\sigma(X, X\sharp\mathbb{P})$.

Method of proof: spike variation

Let $\nu \in L^p(\Omega, \mathcal{F}, \mathbb{P}; U)$ and let $s \in [0, T]$ be a regular point.

Perturbated control:

$$u^h_\nu(t,\omega) \triangleq \begin{cases} u^*(t,\omega), & t \in [0,s), \\ \nu(\omega), & t \in [s, s+h), \\ u^*(t,\omega) & t \in [s+h, T]. \end{cases}$$

Perturbated motion: $Z^h_\nu \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \Gamma)$ satisfies

$$\frac{d}{dt} Z^h_\nu(t,\omega) = f(t, Z^h_\nu(t,\omega), Z^h_\nu(t)\sharp\mathbb{P}, u^h_\nu(t,\omega)),$$

$$Z^h_\nu(0,\omega) = X_0(\omega).$$

Sequence of perturbated controls

There exists a sequence $\{h_n\}_{n=1}^{\infty}$ such that

1. $\{Z_{\nu}^{h_n}(t,\omega)\}_{n=1}^{\infty}$ converges to $X^*(t,\omega)$ for $\lambda \otimes \mathbb{P}$-a.e. $(t,\omega) \in [s,T] \times \Omega$;
2. $\{Z_{\nu}^{h_n}(T,\omega)\}_{n=1}^{\infty}$ converges to $X^*(T,\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Variational equation. Notation

- derivative w.r.t. x:

$$f_x^*(t,\omega) \triangleq \nabla_x f(t, X^*(t,\omega), X^*(t)\sharp\mathbb{P}, u^*(t,\omega));$$

- derivative w.r.t. m:

$$f_m^*(t,\omega,\omega') \triangleq \nabla_m f(t, X^*(t,\omega), X^*(t)\sharp\mathbb{P}, X^*(t,\omega'), u^*(t,\omega));$$

- inner product:

$$\langle f_m^*, Y_\nu\rangle(t,\omega) \triangleq \int_\Omega f_m^*(t,\omega,\omega')Y_\nu(t,\omega')\mathbb{P}(d\omega');$$

- jump:

$$\Delta_\nu^s f^*(\omega) \triangleq f(s, X^*(s,\omega), X^*(s)\sharp\mathbb{P}, \nu(\omega)) \\ - f(s, X^*(s,\omega), X^*(s)\sharp\mathbb{P}, u^*(s,\omega)),$$

Variational equation

$$\frac{d}{dt} Y_\nu(t,\omega) = f_x^*(t,\omega) \cdot Y_\nu(t,\omega) + \langle f_m^*, Y_\nu \rangle(\tau,\omega),$$
$$Y_\nu(s,\omega) = \Delta_\nu^s f^*(\omega).$$

Variational equation

$$\frac{d}{dt} Y_\nu(t,\omega) = f_x^*(t,\omega) \cdot Y_\nu(t,\omega) + \langle f_m^*, Y_\nu \rangle(\tau,\omega),$$
$$Y_\nu(s,\omega) = \Delta_\nu^s f^*(\omega).$$

Proposition (derivative of perturbated motions).

$$\frac{1}{h_n} \| Z_\nu^{h_n}(t) - X^*(t) - h_n Y_\nu(t) \|_{L^p} \to 0 \text{ as } n \to \infty$$

uniformly for $t \in (s, T]$.

Variation of running cost. Notation

- derivative w.r.t. x:

$$f_{0,x}^{*}(t,\omega) \triangleq \nabla_x f_0(t, X^*(t,\omega), X^*(t)\sharp\mathbb{P}, u^*(t,\omega)),$$

- derivative w.r.t. m:

$$f_{0,m}^{*}(t,\omega,\omega') \triangleq \nabla_m f_0(t, X^*(t,\omega), X^*(t)\sharp\mathbb{P}, X^*(t,\omega'), u^*(t,\omega)),$$

- inner product:

$$\langle f_{0,m}^{*}, Y_\nu \rangle(t,\omega) \triangleq \int_\Omega f_{0,m}^{*}(t,\omega,\omega') Y_\nu(t,\omega')\mathbb{P}(d\omega');$$

- jump:

$$\begin{aligned}\Delta_\nu^s f_0^*(\omega) \triangleq{} & f(s, X^*(s,\omega), X^*(s)\sharp\mathbb{P}, \nu(\omega)) \\ & - f_0(s, X^*(s,\omega), X^*(s)\sharp\mathbb{P}, u^*(s,\omega)).\end{aligned}$$

Variation of the running cost

$$\lim_{n\to\infty} \frac{1}{h_n}\Big[\int_0^T \mathbb{E} f_0(t, Z_\nu^{h_n}(t), Z_\nu^{h_n}(t)\sharp\mathbb{P}, u_\nu^{h_n}(t))dt - \int_0^T \mathbb{E} f_0(t, X^*(t), X^*(t)\sharp\mathbb{P}, u^*(t))dt\Big]$$

$$= \mathbb{E}\Delta_\nu^s f_0^* + \int_s^T \mathbb{E}[f_{0,x}^*(t)Y_\nu(t) + \langle f_{0,m}^*, Y_\nu\rangle(t)]dt.$$

Variation of the terminal cost. Notation

- derivative w.r.t. x:

$$\sigma_x^*(\omega) \triangleq \nabla_x \sigma(X^*(T,\omega), X^*(T)\sharp\mathbb{P});$$

- derivative w.r.t. x:

$$\sigma_m^*(\omega,\omega') \triangleq \nabla_m \sigma(X^*(T,\omega), X^*(T)\sharp\mathbb{P}, X^*(T,\omega'));$$

- inner product:

$$\langle \sigma_m^*, Y_\nu \rangle(\omega) \triangleq \int_\Omega \sigma_m^*(\omega,\omega') Y_\nu(\omega')\mathbb{P}(d\omega').$$

Variation of the terminal cost

$$\lim_{n\to\infty} \frac{1}{h_n}\mathbb{E}|\sigma(Z_\nu^{h_n}(T), Z_\nu^{h_n}(T)\sharp\mathbb{P}) - \sigma(X^*(T), X^*(T)\sharp\mathbb{P}) \\ - h_n[\sigma_x^* + \langle \sigma_m^*, Y_\nu\rangle]| = 0.$$

Local minimizers and PMP for the Kantorovich approach

Kantorovich approach

- Control process: (η, u_K), where $\eta \in \mathcal{P}^p(\Gamma)$, $u_K \in L^p(\Gamma, \mathcal{B}(\Gamma), \eta; \mathcal{U}^p)$.
- Feasibility: for η-a.e. $\gamma \in \Gamma$,

$$\frac{d}{dt}\gamma(t) = f(t, \gamma(t), e_t \sharp \eta, u_K(t, \gamma)),$$

  where $e_t(\gamma) = \gamma(t)$, $(e_t \sharp \eta)(E) = \eta\{\gamma \in \Gamma : \gamma(t) \in E\}$.
- Initial condition: $e_0 \sharp \eta = m_0$.
- Payoff:

$$J_K(\eta, u_K) \triangleq \int_\Gamma \sigma(\gamma(T), e_T \sharp \eta)\eta(d\gamma) + \int_\Gamma \int_0^T f_0(t, \gamma(t), e_t \sharp \eta, u_K(t, \gamma))dt\eta(d\gamma).$$

Kantorovich local minimizer

A Kantorovich control process (η^*, u_K^*) is a strong local minimizer at m_0 if there exists $\varepsilon > 0$ such that: for any Kantorovich process (η, u_K) satisfying $W_p(e_t\sharp\eta, e_t\sharp\eta^*) \leq \varepsilon$,

$$J_K(\eta^*, u_K^*) \leq J_K(\eta, u_K).$$

Kantorovich and Lagrangian processes

Let

- (η, u_K) be a Kantorovich control process;
- $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space.

A Lagrangian control process (X, u_L) defined on $(\Omega, \mathcal{F}, \mathbb{P})$ realizes (η, u_K) if

- $\eta = X(\cdot)\sharp\mathbb{P}$,
- $u_L(t, \omega) = u_K(t, X(\cdot, \omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and a.e. $t \in [0, T]$.

Kantorovich minimum implies Lagrangian minimum

Theorem

Assume that (η^*, u_K^*) is a strong local minimizer in the framework of the Kantorovich approach at $m_0 = e_t \sharp \eta^*$. Let (X^*, u_L^*) be a Lagrangian process that realizes the Kantorovich process (η^*, u_K^*). Then, (X^*, u_L^*) is a strong local minimizer at m_0 in the framework of the Lagrangian approach.

Kantorovich minimum implies Lagrangian minimum

Theorem

Let (η, u_K) be a Kantorovich control process. Assume also that the standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is such that either the probability $\mathbb{P}$ has no atoms or $(\Omega, \mathcal{F}, \mathbb{P}) = (\Gamma, \mathcal{B}(\Gamma), \eta)$.
Then, there exists a Lagrangian process (X, u_L) defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that realizes (η, u_K).
Furthermore, if $(\Omega, \mathcal{F}, \mathbb{P}) = (\Gamma, \mathcal{B}(\Gamma), \eta)$, one can put $X = \mathrm{id}_\Omega$ and $u_K = u_L$.

Pontryagin maximum principle in the Kantorovich form

Let (η^*, u_K^*) be a strong local minimizer in the framework of the Kantorovich approach.
Then, there exists a function $\psi \in L^q(\Gamma, \mathcal{B}(\Gamma), \eta^*; \Gamma^*)$ such that

- the costate equation,
- transversality condition,
- maximization conditions

hold true.

Costate equation

For η^*-a.e. $\gamma \in \Gamma$, $\psi(\cdot,\gamma)$ solves

$$\begin{aligned}
&\frac{d}{dt}\psi(t,\gamma) \\
&\quad = -\psi(t,\gamma)\nabla_x f(t,\gamma(t), e_t\sharp\eta^*, u_K^*(t,\gamma)) \\
&\qquad\quad + \nabla_x f_0(t,\gamma(t), e_t\sharp\eta^*, u_K^*(t,\gamma)) \\
&\qquad\quad - \int_\Gamma \psi(t,\gamma')\nabla_m f(t,\gamma'(t), e_t\sharp\eta^*, \gamma(t), u_K^*(t,\gamma'))\eta^*(d\gamma') \\
&\qquad\quad + \int_\Gamma \nabla_m f_0(t,\gamma'(t), e_t\sharp\eta^*, \gamma(t), u_K^*(t,\gamma'))\eta^*(d\gamma').
\end{aligned}$$

Transversality condition

$$\psi(T,\gamma)\sharp\eta^* = -\nabla_x \sigma(\gamma(T), e_T\sharp\eta^*) \\ - \int_\Gamma \nabla_m \sigma(\gamma'(t), e_T\sharp\eta^*, \gamma(T))\eta^*(d\gamma')$$

for η^*-a.e. $\gamma \in \Gamma$.

Maximization of the Hamiltonian condition

$$
\begin{aligned}
H(s, \gamma(s), e_s\sharp\eta^*, &\psi(s,\gamma), u^*(s,\gamma)) \\
&= \max_{u \in U} H(s, \gamma(s), e_s\sharp\eta^*, \psi(s,\gamma), u)
\end{aligned}
$$

for a.e. $s \in [0, T]$ and η^*-a.e. $\gamma \in \Gamma$.

Local minimizers and PMP for the Eulerian approach

Eulerian approach

- Control process: $(m(\cdot), u_E)$, where $m(t)$ is a probability on $\mathbb{R}^d$, $u_E : [0, T] \times \mathbb{R}^d \to U$.
- Dynamics: $m(\cdot)$ is a distributional solution of the nonlocal continuity equation:

$$\partial_t m(t) + \mathrm{div}(v_E(t,x)m(t)) = 0,$$

  for $v_E(t,x) = f(t,x,m(t),u_E(t,x))$.
- Initial condition: $m(0) = m_0$.
- Payoff:

$$J_E(\mu, u_E) \triangleq \int_{\mathbb{R}^d} \sigma(x, m(T))m(T, dx) + \int_0^T \int_{\mathbb{R}^d} f_0(t, x, m(t), u_E(t,x))m(t, dx)dt.$$

Convexity condition for the Eulerian case

- the set U is a closed convex subset of a Banach space;
- the mapping $U \ni u \mapsto f(t,x,m,u)$ is affine in u, i.e., for $t \in [0,T]$, $x \in \mathbb{R}^d$, $m \in \mathcal{P}^p(\mathbb{R}^d)$, $u_1, u_2 \in U$, $\alpha \in [0,1]$,

$$f(t,x,m,\alpha u_1 + (1-\alpha)u_2) \\ = \alpha f(t,x,m,u_1) + (1-\alpha) f(t,x,m,u_2);$$

- the function f_0 is convex in u, i.e., for every $t \in [0,T]$, $x \in \mathbb{R}^d$, $m \in \mathcal{P}^p(\mathbb{R}^d)$, $u_1, u_2 \in U$, $\alpha \in [0,1]$,

$$f_0(t,x,m,\alpha u_1 + (1-\alpha)u_2) \\ \leq \alpha f_0(t,x,m,u_1) + (1-\alpha) f_0(t,x,m,u_2).$$

Eulerian local minimizer

A Eulerian control process $(m^*(\cdot), u_E^*)$ is a strong local minimizer at m_0 if there exists $\varepsilon > 0$ such that: for any Eulerian process $(m(\cdot), u_E)$ satisfying $W_p(m(t), m^*(t)) \leq \varepsilon$,

$$J_E(m^*(\cdot), u_E^*) \leq J_E(m(\cdot), u_E).$$

Eulerian and Lagrangian processes

Let $(m(\cdot), u_E)$ be an Eulerian control process. A Lagrangian control process (X, u_L) defined on a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ realizes $(m^*(\cdot), u_E)$ provided that

- $m(t) = X(t)\sharp\mathbb{P}$ for every $t \in [0, T]$;
- $u_L(t,\omega) = u_E(t, X(t,\omega))$ for a.e. $t \in [0, T]$ and $\mathbb{P}$-a.e. $\omega \in \Omega$.

Eulerian minimizer implies Lagrangian one

Theorem

Let $(m^*(\cdot), u_E^*)$ be a strong local minimizer in the Euler framework and let (X^*, u_L^*) be a Lagrangian process defined on some standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that realizes $(m^*(\cdot), u_E^*)$.
Then, (X^*, u_L^*) is a strong local minimizer at m_0 within the Lagrangian framework.

Eulerian minimizer implies Lagrangian one

Theorem

Assume that $(m(\cdot), u_E)$ *is an Eulerian control process. Furthermore, let a standard probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *be such that at least one the following conditions satisfies:*

- *the probability* $\mathbb{P}$ *has no atoms,*
- $(\Omega, \mathcal{F}, \mathbb{P}) = (\Gamma, \mathcal{B}(\Gamma), \eta)$, *and* η *is concentrated on solutions of*

$$\frac{d}{dt}\gamma(t) = f(t, \gamma(t), m(t), u_E(t, \gamma(t)))$$

  and $m(t) = e_t \sharp \eta$.

Then, there exists a Lagrangian process (X, u_L) *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *that realizes* $(m(\cdot), u_E)$. *Furthermore, in the second case, we can put* $X = \mathrm{id}_\Omega$ *and* $u_E(t, \gamma(t)) = u_L(t, \gamma)$.

Pontryagin maximum principle for the Eulerian framework

Let an Eulerian control process $(m^*(\cdot), u_E^*)$ be a strong local minimizer at m_0. Then, there exists a flow of probabilities $\nu^*(\cdot) : [0, T] \to \mathcal{P}^{p\wedge q}(\mathbb{R}^d \times \mathbb{R}^{d,*})$ satisfying

- consistency condition;
- continuity equation playing both the roles of state and costate equations;
- transversality condition;
- maximization of the Hamiltonian condition.

Consistency condition

$$\mathsf{p}^1 \sharp \nu^*(t) = m^*(t) \qquad \forall t \in [0, T];$$

Here $\mathsf{p}^i(x_1, x_2) \triangleq x_i$.

Joint state and costate continuity equation

$\nu^*(\cdot)$ is a distributional solution of the continuity equation

$$\partial_t \nu^* + \mathrm{div}(\jmath(t,x,\psi)\nu^*) = 0,$$

where the vector field $\jmath(t,x,\psi) = (\jmath_x(t,x,\psi), \jmath_\psi(t,x,\psi))$ is given by

$$\begin{aligned}
\jmath_x(t,x,\psi) &\triangleq f(t,x,m^*(t),u_E^*(t,x)),\\
\jmath_\psi(t,x,\psi) &\triangleq \\
&\quad -\psi\nabla_x f(t,x,m^*(t),u_E^*(t,x))\\
&\qquad + \nabla_x f_0(t,x,m^*(t),u_E^*(t,x))\\
&\qquad - \int_{\mathbb{R}^d\times\mathbb{R}^{d,*}} \zeta\nabla_m f(t,y,m^*(t),x,u_E^*(t,y))\nu^*(t,d(y,\zeta))\\
&\qquad + \int_{\mathbb{R}^d} \nabla_m f_0(t,y,m^*(t),x,u_E^*(t,y))m^*(t,dy);
\end{aligned}$$

Continuity equation

$\nu(\cdot)$ is a distributional solution of the nonlocal continuity equation:

$$\partial_t \nu(t) + \operatorname{div}(\mathcal{J}(t,x,\psi)\nu(t)) = 0,$$

iff, for every $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^{d,*})$,

$$\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^{d,*}} \big[\partial_t\varphi(t,x,\psi) + \nabla_x\varphi(t,x,\psi)\mathcal{J}_x(t,x,\psi) + \mathcal{J}_\psi(t,x,\psi)\nabla_\psi\varphi(t,x,\psi)\big]\nu(t,d(x,\psi))dt = 0.$$

Transversality condition

$$\mathsf{p}^2 \sharp \nu^*(T) = \Big[-\nabla_x \sigma(\cdot, m^*(T)) \\ - \int_{\mathbb{R}^d} \nabla_m \sigma(y, m^*(T), \cdot) m^*(T, dy) \Big] \sharp m^*(T).$$

Here $\mathsf{p}^i(x_1, x_2) \triangleq x_i$.

Maximization condition

$$H(s, x, m^*(s), \psi, u_E^*(s, x)) = \max_{u \in U} H(s, x, m^*(s), \psi, u)$$

for almost every $s \in [0, T]$ and $\nu^*(s)$-a.e. $(x, \psi) \in \mathbb{R}^d \times \mathbb{R}^{d,*}$.

Costate equation in the Hamiltonian form

- Hamiltonian for the Eulerian framework:

$$\mathscr{H}(t,\nu,u) \triangleq \int_{\mathbb{R}^d\times\mathbb{R}^{d,*}} H(t,x,\mathrm{p}^1\sharp\nu,\psi,u)\nu(d(x,\psi)).$$

- Unit symplectic matrix: $\mathbb{J}(\zeta,y)\triangleq(y,-\zeta)$.
- Averaged terminal payoff:

$$\mathcal{S}(m)\triangleq\int_{\mathbb{R}^d}\sigma(x,m)m(dx).$$

State+costate equation:

$$\begin{aligned}
&\partial_t\nu^* + \operatorname{div}(\mathbb{J}\nabla_\nu\mathscr{H}(t,\nu^*(t),x,\psi,u_E^*(t,x))\nu^*) = 0,\\
&\mathrm{p}^1\sharp\nu^*(0) = m_0,\\
&\mathrm{p}^2\sharp\nu^*(T) = -\nabla_m\mathcal{S}(\mathrm{p}^1\sharp\nu^*(T),\cdot)\sharp(\mathrm{p}^1\sharp\nu^*(T)).
\end{aligned}$$

Example. Mean field type linear-quadratic regulator

Dynamics and payoffs

Dynamics of each agent:

$$\frac{d}{dt}X(t,\omega) = A(t)X(t,\omega) + B(t)u(t,\omega),$$

Total payoff:

$$\frac{1}{2}\mathbb{E}\bigg(\int_0^T [X^T(t)Q_x(t)X(t) + u^T(t)R(t)u(t)]dt + X^T(T)K_x(t)X(T) \bigg) + \frac{1}{2}\int_0^T \mathbb{E}[(X(t) - \mathbb{E}X(t))^T Q_m(t)(X(t) - \mathbb{E}X(t))]dt + \frac{1}{2}\mathbb{E}[(X(T) - \mathbb{E}X(T))^T K_m(X(T) - \mathbb{E}X(T))].$$

Optimal control

$$u^*(t,\omega) = -R^{-1}(t)B^T(t)\Big[P_1(t)(X^*(t,\omega) - \mathbb{E}X^*(t)) \\ + P_2(t)\mathbb{E}X^*(t)\Big],$$

Ricatti equation for P_1

$$\frac{d}{dt}P_1(t) = -P_1(t)A(t) - A^T(t)P_1(t) + P_1(t)B(t)R^{-1}(t)B(t)P_1(t) - (Q_x(t) + Q_m(t));$$

boundary condition:

$$P_1(T) = K_x + K_m,$$

Ricatti equation for P_2

$$\frac{d}{dt}P_2(t) = -P_2(t)A(t) - A^T(t)P_2(t) + P_2(t)B(t)R^{-1}(t)B(t)P_2(t) - Q_x(t);$$

boundary condition:

$$P_2(T) = K_x.$$

Thank you for your attention!